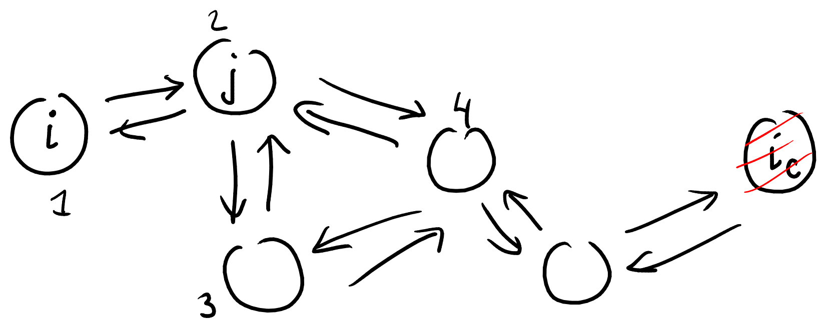
Calculate mean capture time:

model w/ states i + transition matrix Ω_{ij}

$i \neq j$: $=$ prob. to go from j to i per unit time

T_i = avg. time to reach some target state i_c from i for the first time



$T_{i \to i_c}$

$= T_i$ for simplicity

will prove: T_i satisfies a set of equations

N-1 equ's: $$\sum_i T_i \Omega_{ij} = -1 \quad \text{for any } j \neq i_c$$

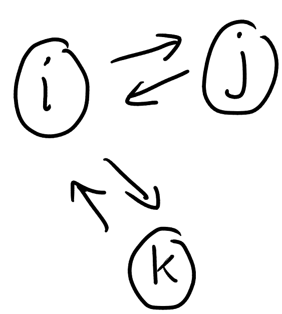
1 equ: $$T_{i_c} = 0$$

N equ's $\Longrightarrow$ N unknowns: $T_i \quad i = 1, \ldots, N$

$\uparrow$ # states

convert to continuum approx: diff. equ for $T(x)$

3 parts: 1)



how long on avg. to leave state i $\equiv T_i^{esc}$

2) find prob. that we went to a specific neighbor j after leaving i $\equiv \Pi_{ji}$

3) construct a recursion equ. to find τ_i using 1) + 2)

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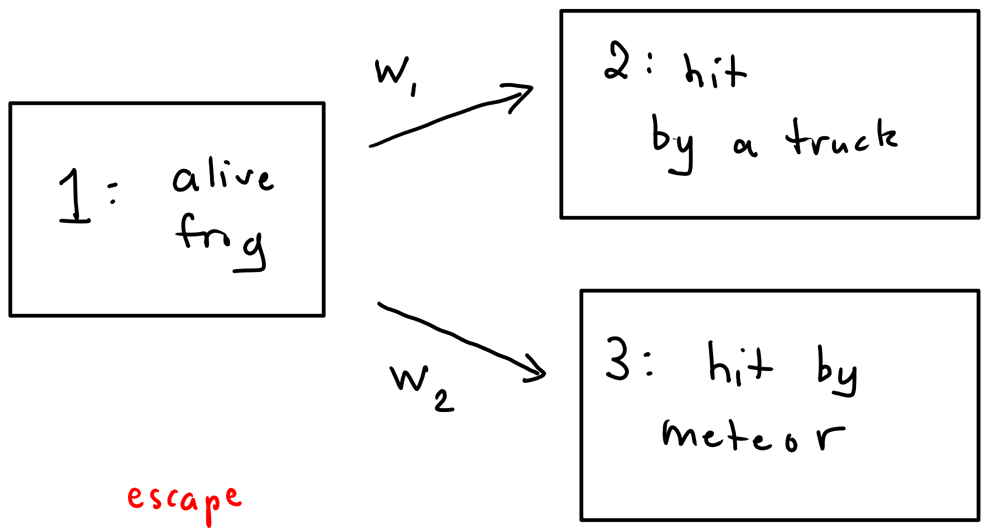
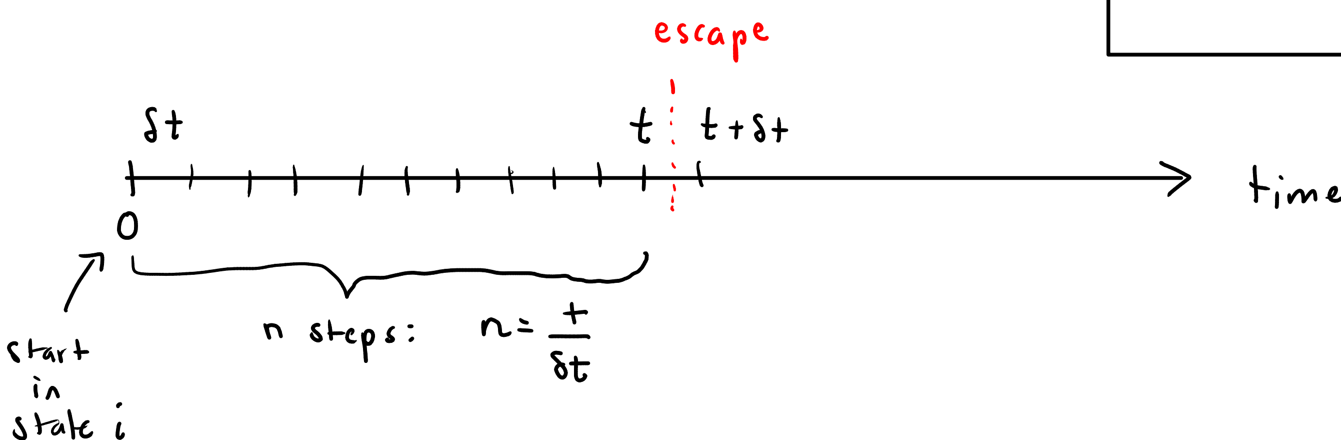
1) example:

$\delta t = 1s$

$w_1 \delta t = 0.05$

$w_2 \delta t = 10^{-20}$





prob. of not leaving state i in time step δt $= 1 - (w_1 + w_2)\delta t$

in general $= 1 - \sum_{j \neq i} \Omega_{ji} \delta t$

$= 1 - |\Omega_{ii}| \delta t$

columns of Ω sum to zero

$$\Omega_{ii} = - \sum_{j \neq i} \Omega_{ji}$$

$$\underbrace{\left[1-|\Omega_{ii}|\delta t\right]^{n=\frac{t}{\delta t}}}_{\substack{\text{prob. to} \\ \text{survive } n \text{ steps}}} \underbrace{|\Omega_{ii}|\delta t}_{\substack{\text{prob. to} \\ \text{escape in} \\ \text{the next} \\ \text{time step}}} = f_i(t)\,\delta t$$

prob. that you escape i b/t time t & $t+\delta t$

$\delta t = \frac{t}{n}$ small

$\approx dt$

$$\cong \exp\left(-|\Omega_{ii}|t\right)|\Omega_{ii}|\,\delta t$$

$$f_i(t) = e^{-|\Omega_{ii}|t}\,|\Omega_{ii}|$$

$$\int_0^\infty dt\, f_i(t) = 1$$

τ_i^{esc} = avg. time to escape i

$$= \int_0^\infty dt\; t\, f_i(t) \simeq \frac{1}{|\Omega_{ii}|}$$

frogger:

$$\Omega = \begin{array}{c|c} & \text{start} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{end} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -w_1 - w_2 & 0 & 0 \\ w_1 & 0 & 0 \\ w_2 & 0 & 0 \end{pmatrix} \end{array}$$

$$\tau_1^{esc} = \frac{1}{|\Omega_{11}|} = \frac{1}{w_1+w_2} \approx \frac{1}{.05} = 20 \text{ s}$$

2) After escaping, which state are we in?

frog: prob. of dying by truck $= \dfrac{w_1\,\delta t}{w_1\,\delta t + w_2\,\delta t} = \dfrac{w_1}{w_1+w_2}$

$\approx 1 - 10^{-20}$

'' ''

dying by meteor $= \dfrac{w_2}{w_1 + w_2}$

$\approx 10^{-20}$

in general: $\pi_{ji} = \dfrac{\Omega_{ji}}{\sum_{k \neq i} \Omega_{ki}} = \dfrac{\Omega_{ji}}{|\Omega_{ii}|}$

outcome (j) start (i)

3) recursion argument:

$i \neq i_c$: $\quad \tau_i = \tau_i^{esc} + \sum_{j \neq i} \pi_{ji} \tau_j$

τ_i: $i \to i_c$

τ_i^{esc}: escape from i

π_{ji}: prob. to land in j

τ_j: how long from $j \to i_c$

$$\tau_i = \frac{1}{-\Omega_{ii}} + \sum_{j \neq i} \frac{\Omega_{ji}}{(-\Omega_{ii})} \tau_j$$

multiply by $+\Omega_{ii}$: $\quad \Omega_{ii} \tau_i = -1 - \sum_{j \neq i} \Omega_{ji} \tau_j$

$$\sum_{j=1}^{N} \Omega_{ji} \tau_j = -1$$

$$\Rightarrow \boxed{\begin{gathered} \sum_j \tau_j \Omega_{ji} = -1 \\ \tau_{i_c} = 0 \end{gathered}}$$

valid for all $i \neq i_c$

example:

$$1 \underset{k_u}{\overset{k_s}{\rightleftarrows}} 2 \xrightarrow{k_b} 3$$

$i_c = 3$ target

$$\Omega = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -k_s & k_u & 0 \\ k_s & -k_u - k_b & 0 \\ 0 & k_b & 0 \end{pmatrix} \end{array}$$

$i = 1$: $$\sum_j \Omega_{j1} \tau_j = -1 \Rightarrow -k_s \tau_1 + k_s \tau_2 = -1$$

$i = 2$: $$\sum_j \Omega_{j2} \tau_j = -1 \Rightarrow k_u \tau_1 - (k_u + k_b)\tau_2 + k_b \tau_3 = -1$$

$$\tau_3 = 0$$

solve: $$\tau_1 = \frac{k_s + k_u + k_b}{k_s k_b} \qquad 1 \to 3$$

$$1 \underset{k_u}{\overset{k_s}{\rightleftarrows}} 2 \xrightarrow{k_b} 3$$

$i_c = 3$ target

$k_s, k_b \to \infty$

$\tau_1 \to 0$ ✓

$k_b \to 0$

$\tau_1 \to \infty$ ✓