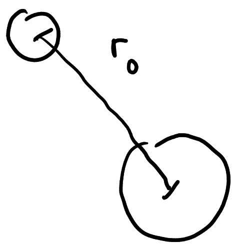
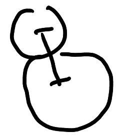
τ_i = mean time from $i \to i_c$

$$i \neq i_c: \quad \sum_{j=1}^{N} \tau_j \Omega_{ji} = -1 \qquad \tau_{i_c} = 0$$

original motivation:

target: "capture" radius





$R = R_1 + R_2$

3D diffusion

initial

separation has

$D = D_1 + D_2$

1D version: continuum approx

$$p_i(t) \to p(x,t)$$

$$\frac{dp_i}{dt} = \sum_j \Omega_{ij} p_j \longrightarrow \frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}$$

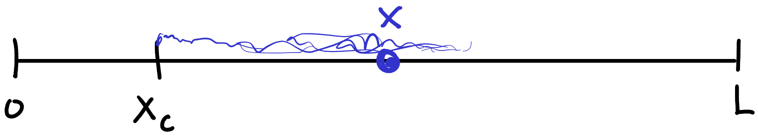
$$\text{b.c.} \quad \left.\frac{\partial p}{\partial x}\right|_{0,L} = 0$$

Same approach: $\tau_i \to \tau(x)$ = avg. time to go from $x \to x_c$

$$i_c \to x_c$$

$$i \neq i_c: \quad \sum_j \tau_j \Omega_{ji} = -1 \Rightarrow D \frac{\partial^2 \tau}{\partial x^2} = -1$$

$$\tau_{i_c} = 0 \qquad \Rightarrow \qquad \tau(x_c) = 0$$



$$\left.\frac{\partial \tau}{\partial x}\right|_L = 0$$

solve:

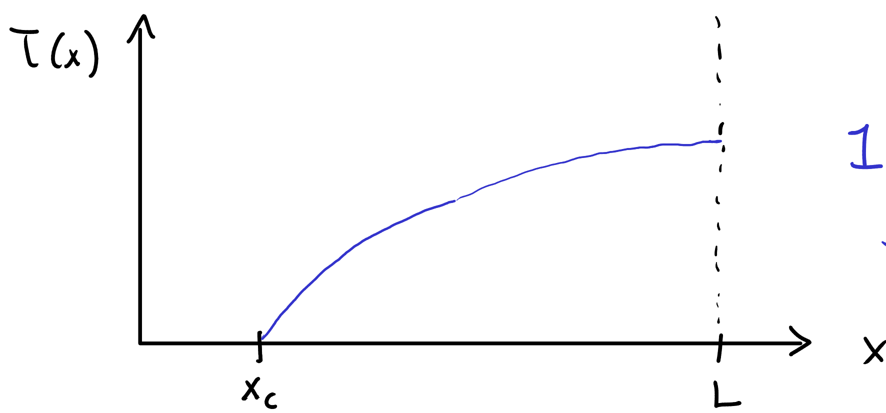
integrate $\Rightarrow$ $$\frac{\partial \tau}{\partial x} = -\frac{x}{D} + C_1$$

integrate $\Rightarrow$ $$\tau = -\frac{x^2}{2D} + C_1 x + C_2$$

$$\left.\frac{\partial \tau}{\partial x}\right|_L = 0 = -\frac{L}{D} + C_1 \Rightarrow C_1 = \frac{L}{D}$$

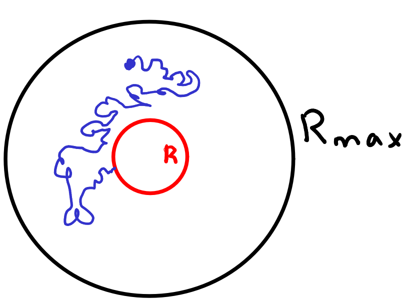
$$\tau(x_c) = 0 \Rightarrow C_2 = \frac{x_c^2}{2D} - \frac{L x_c}{D}$$

$$\Rightarrow \quad \tau(x) = \frac{(x - x_c)(2L - x - x_c)}{2D}$$



1D diffusion to capture

3D: $\tau(\vec{r})$ = mean time to go from separation $\vec{r}$ to some separation $\vec{r}_c$ where $|\vec{r}_c| = R_1 + R_2$
$\equiv R$



$= \tau(r)$ by symmetry

where $r = |\vec{r}|$

Sidney Redner
First passage problems

$$D\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\tau(\vec{r})=-1$$

$$\tau(\vec{r}\in \text{sphere of radius } R)=0$$

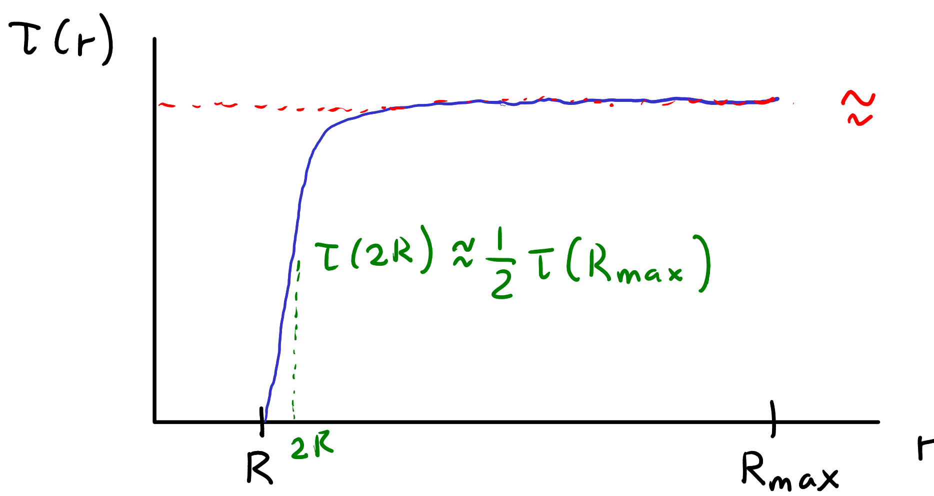
$$\nabla\tau(\vec{r})\Big|_{\vec{r}\in\text{outer boundary}}=0$$

$$\vec{r}=(r,\theta,\phi) \qquad \tau(\vec{r})=\tau(r)$$

$$\Rightarrow \quad D\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right]\tau(r)=-1$$

$$\tau(R)=0 \qquad \frac{\partial\tau}{\partial r}\Big|_{R_{max}}=0$$

$$\Rightarrow \quad \tau(r)=\frac{R_{max}^3}{3DR}-\frac{R_{max}^3}{3Dr}-\frac{r^2}{6D}+\frac{R^2}{6D}$$



$$\approx \frac{R_{max}^3}{3DR} \cong \tau(R_{max})$$

to a good approx: time for two molecules to meet in 3D (in a volume of size $\sim R_{max}^3$)

L^2

$= 3D$

MSD

$= 6Dt$

$$\tau \simeq \frac{R_{max}^3}{3DR}$$ (valid when initial sep > a few nm)

$$= \underbrace{\left(\frac{R_{max}^2}{6D}\right)}_{\sim \text{ avg. time for the sep. to cover dist } R_{max} \text{ (i.e. visit the entire cell volume)}} \underbrace{\left(\frac{2R_{max}}{R}\right)}_{\text{additional factor required for collision to occur}}$$

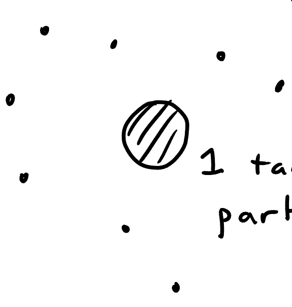
typical protein $R_1 = R_2 \sim 1$ nm

$R_{max} \sim 1\ \mu m$ for bacteria

$$\tau = (8.3 \times 10^{-3}\ s)(1000)$$

$$= 8.3\ s$$

speed up $\Rightarrow$ have more particles



n "searcher" particles

1 target particle

$$\tau = \underbrace{\left(\frac{R_{max}^2}{6D}\right)}_{\text{time scale to explore the cell}} \left(\frac{2R_{max}}{Rn}\right)$$

prob. of collision has increased by a factor of n

$$V = \frac{4}{3}\pi R_{max}^3$$ (sph. volume)

$$\Rightarrow \quad \tau = \frac{V}{4\pi D R n} = \frac{1}{4\pi D R c}$$

$c = \frac{n}{V}$

= conc. of searcher particles

rate of searchers hitting target

$$k = \frac{1}{\tau} = \boxed{4\pi D R c \equiv k_{smol}}$$

Smoluchowski rate

"speed limit" for any chemical reaction that requires 3D diffusion

any reaction rate

$$k \le k_{smol.}$$