

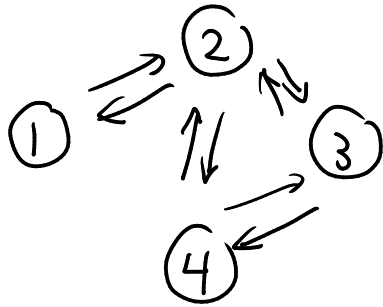
Proof:

ingredients:

- any system described by time-indep. trans. matrix Ω

- transition graph associated w/ Ω is connected

(path via graph b/t any two states)



Set up parallels to quantum mechanics:

	quantum	master equ.	
description of state	$ \psi(t)\rangle$ vector in Hilbert space	$\vec{P}(t)$ vector in prob. space	w/ comps $P_n(t)$
dynamical eqn.	$i\hbar \frac{\partial}{\partial t} \psi(t)\rangle = \hat{H} \psi(t)\rangle$	$\frac{\partial}{\partial t} \vec{P}(t) = \Omega \vec{P}(t)$	
solution (time-ind. $\hat{H}$ or Ω)	$ \psi(t)\rangle = \underbrace{e^{-\frac{i\hat{H}t}{\hbar}}}_{\text{propagator}} \psi(0)\rangle$	$\vec{P}(t) = \underbrace{e^{\Omega t}}_{\text{propagator}} \vec{P}(0)$	
avg. of observable	$A(t) = \langle \psi(t) \hat{A} \psi(t) \rangle$ $= \langle \psi(0) \hat{A}^H(t) \psi(0) \rangle$ $\uparrow$ Heisenberg oper.	$A(t) = \vec{\hat{A}} \cdot \vec{P}(t)$	
Heisenberg dyn. eqn.	$\frac{d}{dt} \hat{A}^H(t) = \frac{i}{\hbar} [\hat{H}, \hat{A}^H(t)]$	$\frac{\partial}{\partial t} \vec{\hat{A}}^H(t) = \vec{\hat{A}}^H(t) \Omega$	

Claim: $\vec{p}(t) = e^{\Omega t} \vec{p}(0)$ is a solution to master eqn.

$$e^{\Omega t} \equiv \mathbb{I} + \Omega t + \frac{\Omega^2 t^2}{2!} + \frac{\Omega^3 t^3}{3!} + \dots$$

identity matrix

def'n of matrix exp.

$$\begin{aligned} \frac{d}{dt} e^{\Omega t} &= 0 + \Omega + \Omega t + \frac{\Omega^3 t^2}{2!} + \dots \\ &= \Omega \left(\mathbb{I} + \Omega t + \frac{\Omega^2 t^2}{2!} + \dots \right) \\ &= \left(\mathbb{I} + \Omega t + \frac{\Omega^2 t^2}{2!} + \dots \right) \Omega \\ &= \Omega e^{\Omega t} = e^{\Omega t} \Omega \end{aligned}$$

plug into master eq. $\frac{d}{dt} \vec{p}(t) = \Omega \vec{p}(t)$

$$\frac{d}{dt} \underbrace{(e^{\Omega t} \vec{p}(0))}_{\vec{p}(t)} = \Omega \underbrace{e^{\Omega t} \vec{p}(0)}_{\vec{p}(t)}$$

✓
satisfies
master
eqn.

$$\vec{p}(t) = \underbrace{e^{\Omega t}}_{\text{propagator}} \vec{p}(0)$$

propagator = transforms $\vec{p}(0)$ into $\vec{p}(t)$

classical averages of observables:

state property $A_n \Rightarrow$ comp. of vector $\vec{A}$

$$A(t) = \sum_n A_n p_n(t)$$

$$= \vec{A} \cdot \vec{p}(t)$$

Heisenberg picture: $|\psi(t)\rangle = \underbrace{\hat{U}(t)}_{\text{propagator}} |\psi(0)\rangle$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

$$A(t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle \quad \langle \psi(t) | = \langle \psi(0) | \hat{U}^\dagger(t)$$

$$= \langle \psi(0) | \underbrace{\hat{U}^\dagger(t) \hat{A} \hat{U}(t)}_{\equiv \hat{A}^H(t)} | \psi(0) \rangle$$

"classical" Heisenberg picture:

$$\vec{p}(t) = e^{\Omega t} \vec{p}(0)$$

$$A(t) = \sum_n A_n p_n(t)$$

$$p_n(t) = (e^{\Omega t} \vec{p}(0))_n$$

$$= \sum_n A_n (e^{\Omega t} \vec{p}(0))_n$$

$$= \sum_n \sum_m A_n (e^{\Omega t})_{nm} p_m(0)$$

$$\equiv \sum_m A_m^H(t) p_m(0)$$

class.
Heis.
observable

$$\hookrightarrow A_m^H(t) \equiv \sum_n A_n (e^{\Omega t})_{nm}$$

vector form: $\vec{A}^H(t) = \vec{A}^T e^{\Omega t}$

$$\frac{d}{dt} \vec{A}^H(t) = \vec{A}^T \frac{d}{dt} e^{\Omega t} = \underbrace{\vec{A}^T e^{\Omega t}}_{\vec{A}^H(t)} \Omega$$

$$\Rightarrow \text{Heisenberg dynamical} \quad \frac{d}{dt} \vec{A}^H(t) = \vec{A}^H(t) \Omega$$

equ.
(adjoint master equ.)

Write out adjoint master equ. in component form:

$$\frac{d}{dt} \vec{A}^H(t) = \vec{A}^H(t) \Omega$$

$$\frac{d}{dt} A_n^H(t) = \sum_m A_m^H(t) \Omega_{mn}$$