

observable $A_n \Rightarrow \vec{A}$

$$A(t) = \sum_n A_n p_n(t) = \vec{A} \cdot \vec{p}(t) \\ = \vec{A}^H(t) \cdot \vec{p}(0)$$

$$\vec{A}^H(t) = \vec{A}^T e^{\Omega t}$$

satisfies: $\frac{d}{dt} \vec{A}^H(t) = \vec{A}^H(t) \Omega$

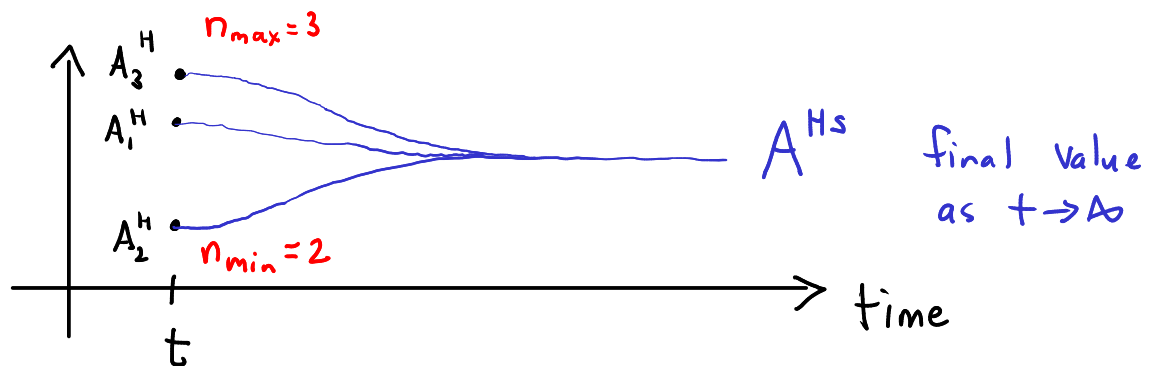
Write it in comp. form: $\frac{d}{dt} A_n^H(t) = \sum_m A_m^H(t) \Omega_{mn}$

$$= \sum_{m \neq n} A_m^H \Omega_{mn} + A_n^H \Omega_{nn}$$

$$= \sum_{m \neq n} A_m^H \Omega_{mn} + A_n^H \left(- \sum_{m \neq n} \Omega_{mn} \right)$$

$$\Rightarrow \frac{d}{dt} A_n^H(t) = \sum_{m \neq n} (A_m^H(t) - A_n^H(t)) \Omega_{mn}$$

at time t
current
values are
 $A_n^H(t)$



at any
time we can define:

$n = n_{\max}$ has largest $A_n^H(t)$
 $n = n_{\min}$ has smallest $A_n^H(t)$

$A_{n_{\max}}^H$
is
always
decreasing

$$\frac{d}{dt} A_{n_{\max}}^H = \sum_{m \neq n_{\max}} \underbrace{(A_m^H - A_{n_{\max}}^H)}_{< 0} \underbrace{\Omega_{mn_{\max}}}_{\text{at least one } m \text{ has to have } \Omega_{mn_{\max}} > 0 \text{ b/c we have a connected graph}} < 0$$

$A_{n_{\min}}^H$
is
always
increasing

$$\frac{d}{dt} A_{n_{\min}}^H = \sum_{m \neq n_{\min}} \underbrace{(A_m^H - A_{n_{\min}}^H)}_{> 0} \Omega_{mn_{\min}} > 0$$

For a connected graph: $A_n^H(t) \xrightarrow{t \rightarrow \infty} A^{Hs}$ for all n

$$\underbrace{A(t)}_{\text{physical observ. average}} = \sum_m A_m^H(t) p_m(0) \xrightarrow{t \rightarrow \infty} A^{Hs} \underbrace{\sum_m p_m(0)}_1 = A^{Hs}$$

no dependence on $p_m(0)$

This works for any state observable:

construct our own to complete proof.

define "indicator"
observable

$$A_n^{(k)} = \delta_{nk} = \begin{cases} 1 & \text{if } n=k \\ 0 & \text{if } n \neq k \end{cases}$$

$$\text{avg. of } A_n^{(k)} \Rightarrow A^{(k)}(t) = \sum_n A_n^{(k)} p_n(t) \\ = p_k(t)$$

$$\text{theorem} \Rightarrow \begin{array}{ccc} A^{(k)}(t) & \xrightarrow{t \rightarrow \infty} & A^{(k)Hs} \\ \parallel & & \parallel \\ p_k(t) & \xrightarrow{t \rightarrow \infty} & p_k^s \end{array} \begin{array}{l} \text{limiting} \\ \text{value} \\ \\ \text{stationary} \\ \text{probability} \end{array}$$

For a connected graph w/ time-ind. Ω
no matter initial state $\vec{p}(0)$ we
always go to unique final stationary
state $\vec{p}^s$.

Consequences: combined 1st + 2nd law
of thermodynamics

$$\dot{W}(t) - T \dot{I}(t) = \dot{F}(t) = \frac{d}{dt} F(t)$$

as $t \rightarrow \infty$:

$$\dot{W} - T \dot{I} \rightarrow 0$$

$\uparrow$
net work
rate done on
sys.

< 0 or > 0

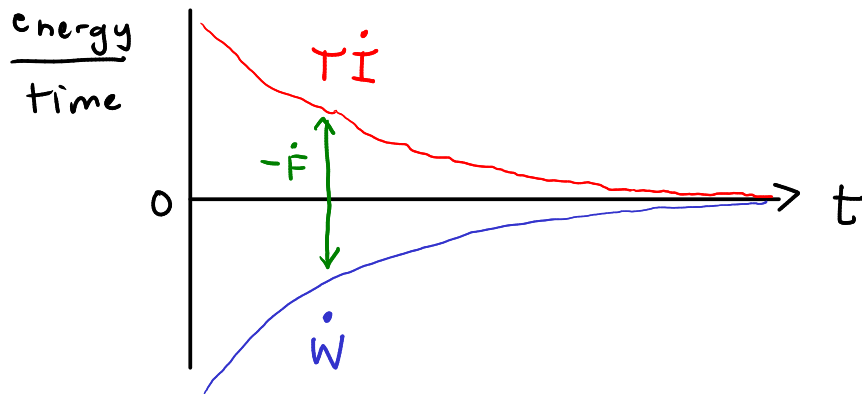
$$F(t) = E(t) - TS(t)$$

$\uparrow \quad \uparrow$
funcs. of $\vec{p}(t)$

$$\Rightarrow \dot{F}(t) \xrightarrow{t \rightarrow \infty} 0$$

Consider scenarios:

case I: $\dot{W} < 0$ sys does net work on env.



as $t \rightarrow \infty$

$\dot{W} \rightarrow 0$ from below

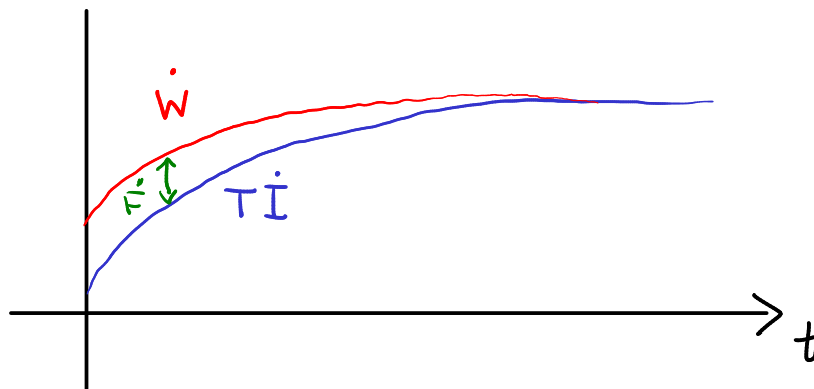
$T\dot{I} \rightarrow 0$ from above

$\dot{F} \rightarrow 0$

b/c $\dot{I} \rightarrow 0 \Rightarrow$ ESS for case I

case II: $\dot{W} > 0$ as $t \rightarrow \infty$

(continuous supply of work from outside, w/ net work on sys.)



$\dot{W} > 0$ as $t \rightarrow \infty$

$\dot{I} > 0$

$\dot{W} - T\dot{I} \rightarrow 0$

$\dot{F} \rightarrow 0$

case II $\Rightarrow \dot{I} > 0$

NESS

What is $\dot{W}$?

$\dot{W} = P_{in} - P_{out}$
 net power into sys input power (all pos. terms) output power (all neg. terms)

as $t \rightarrow \infty$: $\underbrace{P_{in} - P_{out}}_{P_{diss}} = T \dot{I} > 0$

= power dissipated
as heat into env.

to get NESS $\Rightarrow$ need to waste energy

perfect conversion: $P_{out} = P_{in} \Rightarrow \dot{I} = 0$
 (boring) $\Rightarrow$ ESS

Anything interesting in biology: $P_{diss} > 0$
 $(\dot{I} > 0)$