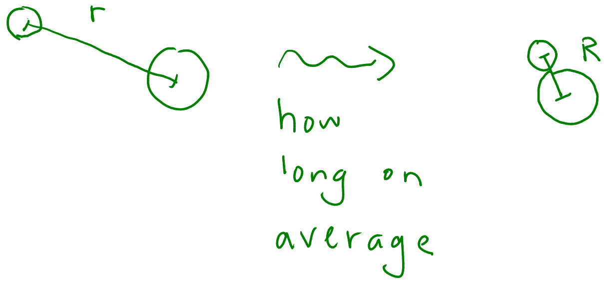
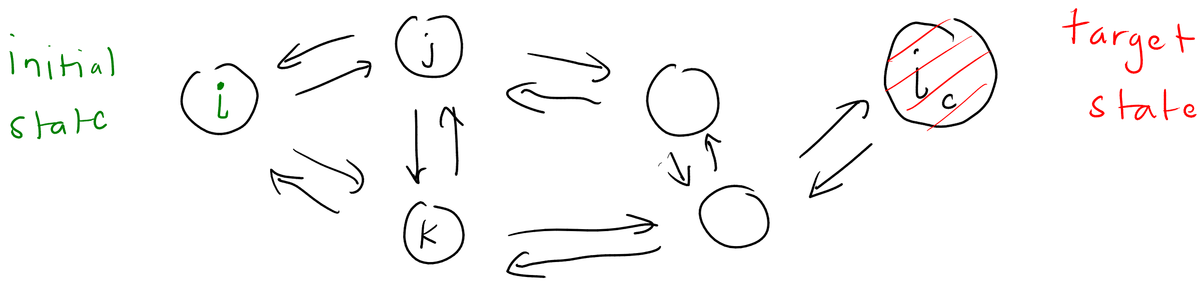


GOAL: calculate $\tau(r_0)$ = mean time to first contact starting at sep. r_0

SETUP: start w/ general systems w/ states i + transition matrix Ω_{ij}



problem: find τ_i = avg. time to reach i_c from i for the first time

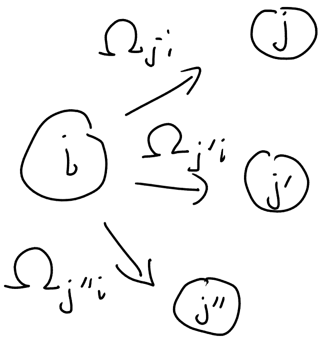
SOLUTION in 3 parts:

1) how long on avg. to escape i ; τ_i^{esc}

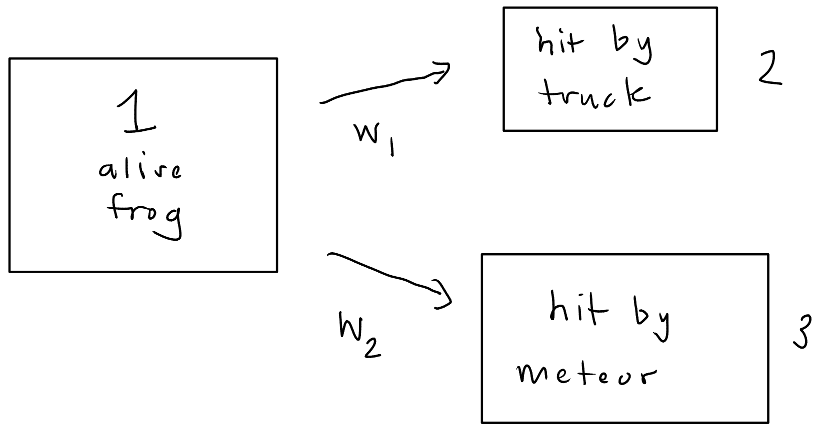
2) find prob. that we escaped to neigh. j after leaving $\equiv \Pi_{ji}$

3) construct a recursive argument to find τ_i using #1 + #2

part 1:

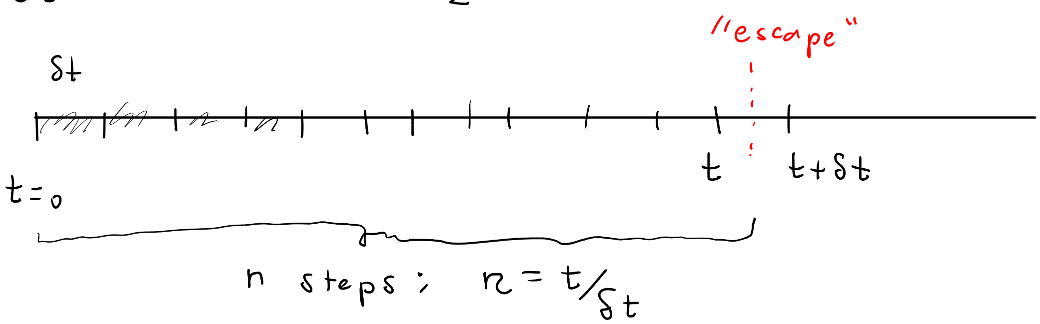


concrete example:



$\delta t = 1\,s$

$w_1 \delta t = 0.05 \qquad w_2 \delta t = 10^{-20}$



prob. of not leaving state i in time δt $= 1 - (w_1 + w_2)\,\delta t$

$$= 1 - \sum_{j \neq i} \Omega_{ji}\,\delta t$$

sum of all outgoing arrows

columns of Ω sum to zero: $\Omega_{ii} = -\sum_{j \neq i} \Omega_{ji}$

$$= 1 - |\Omega_{ii}|\,\delta t$$

$f_i(t)\,\delta t$ = prob. that you escape i between times t & $t + \delta t$

$$= \underbrace{\left[1 - |\Omega_{ii}|\,\delta t\right]^n}_{\text{prob. to survive n steps}} \underbrace{|\Omega_{ii}|\,\delta t}_{\text{prob. to escape in nth step}}$$

$\delta t = \dfrac{t}{n}$

$n \to \infty \quad \delta t \to 0$

$$\approx \exp(-|\Omega_{ii}| t) \, |\Omega_{ii}| \, \delta t$$

$$\Rightarrow \quad f_i(t) = |\Omega_{ii}| e^{-|\Omega_{ii}| t}$$

$$\int_0^\infty dt \; f_i(t) = 1$$ properly normalized prob.

τ_i^{esc} = avg. escape time from i

$$= \int_0^\infty dt \; t \, f_i(t) = \frac{1}{|\Omega_{ii}|}$$

frog: $$\Omega = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{pmatrix} -w_1-w_2 & 0 & 0 \\ w_1 & 0 & 0 \\ w_2 & 0 & 0 \end{pmatrix}$}} \\ 2 & & & \\ 3 & & & \end{array}$$

$$\tau_1^{esc} = \frac{1}{|\Omega_{11}|} = \frac{1}{w_1 + w_2}$$

$$\approx \frac{1}{.05} = 20 \text{ s}$$

2) After escaping, which state are we in?

frog: prob. of escaping to 2 $= \dfrac{w_1}{w_1 + w_2} \approx 1 - 10^{-20}$

" " to 3 $= \dfrac{w_2}{w_1 + w_2} \approx 10^{-20}$

in general: $$\underset{\substack{\uparrow \;\; \uparrow \\ \text{outcome} \;\; \text{start}}}{\pi_{ji}} = \frac{\Omega_{ji}}{\sum_{k \neq i} \Omega_{ki}} = \frac{\Omega_{ji}}{|\Omega_{ii}|}$$

3) recursion argument:

τ_i = avg. time to go $i \to i_c$ for the first time

$i \neq i_c$:
$$\underset{i \to i_c}{\tau_i} = \underset{\substack{\text{escape} \\ \text{from } i}}{\tau_i^{esc}} + \sum_{j \neq i} \underset{\substack{\uparrow \\ \text{prob. to} \\ \text{land in} \\ j}}{\Pi_{ji}} \underset{j \to i_c}{\tau_j}$$

$i = i_c$: $\tau_{i_c} = 0$

$$-\sum_{j \neq i} \frac{\Omega_{ji} \tau_j}{\Omega_{ii}}$$

$$\tau_i = -\frac{1}{\Omega_{ii}} + \sum_{j \neq i} \frac{\Omega_{ji}}{(-\Omega_{ii})} \tau_j = \frac{-1}{\Omega_{ii}} \sum_{j \neq i} \Omega_{ji} \tau_j$$

$$\Omega_{ii} \tau_i = -1 - \sum_{j \neq i} \Omega_{ji} \tau_j$$

$$\Rightarrow \boxed{\sum_{j=1}^{N} \Omega_{ji} \tau_j = -1 \quad i \neq i_c} \qquad \boxed{\tau_{i_c} = 0}$$

N−1 equ. 1 eq $\Rightarrow$ N equ's for N unknowns τ_i, $i = 1, 2, \dots, N$

separate equ. for each choice of i

example:

$$\begin{tikzcd} ① \arrow[r, shift left, "k_s"] & ② \arrow[l, shift left, "k_u"] \arrow[r, "k_b"] & ③ \end{tikzcd}$$

target: $i_c = 3$

$$\boxed{\tau_3 = 0}$$

$$\Omega = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -k_s & k_u & 0 \\ k_s & -k_u - k_b & 0 \\ 0 & k_b & 0 \end{pmatrix} \end{array}$$

$i = 1$: $$\sum_j \Omega_{j1} \tau_j = -1 \Rightarrow \Omega_{11} \tau_1 + \Omega_{21} \tau_2 + \Omega_{31} \tau_3 = -1$$

$$\boxed{-k_s \tau_1 + k_s \tau_2 = -1}$$

$i = 2$: $$\sum_j \Omega_{j2} \tau_j = -1 \Rightarrow \boxed{k_u \tau_1 - (k_u + k_b) \tau_2 + k_b \tau_3 = -1}$$

Solution: $\tau_1 = \dfrac{k_s + k_u + k_b}{k_s k_b}$ avg. time from 1 to 3