Simple interaction w/ env: measurements

ensemble: frac. p_1 (0.2) p_2 (0.1) ....

states $|\psi_1\rangle$ $|\psi_2\rangle$ ....

measure $\hat{A}$ w/ e-states

$\hat{A}|a\rangle = a|a\rangle$

after meas. collapse to one of $|a\rangle$ w/ prob. $|\langle a|\psi_1\rangle|^2$

" " $|a\rangle$ w/ prob. $|\langle a|\psi_2\rangle|^2$

ensemble after meas.

frac. p_a p_b ...

state $|a\rangle$ $|b\rangle$ ... ⇐ e-states of $\hat{A}$

$$p_a = \sum_n p_n |\langle a|\psi_n\rangle|^2$$

p_n: frac. of $|\psi_n\rangle$ in orig. ensemble; $|\langle a|\psi_n\rangle|^2$: prob. $|\psi_n\rangle$ collapses to $|a\rangle$

new dens.

$$\hat{\rho}' = \sum_a p_a |a\rangle\langle a|$$

$$= \sum_{a,n} p_n \langle a|\psi_n\rangle\langle\psi_n|a\rangle |a\rangle\langle a|$$

$$= \sum_a |a\rangle\langle a| \left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right] |a\rangle\langle a|$$

$\hat{P}_a = |a\rangle\langle a|$

$\hat{P}_a^\dagger = |a\rangle\langle a|$

$\hat{P}_a$: projection operator for state $|a\rangle$

$\left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right]$: orig. density operator $\hat{\rho}$

$$\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger$$

density operator transform under meas.

vs.

$$\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^\dagger$$

dens. operator transform for isolated qu. systems

example: initially 100% in state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\hat{\rho} = |\psi_1\rangle\langle\psi_1|$$

ON decomp $\Rightarrow$ basis where $\hat{\rho}$ is diag.

$$= \{ |\psi_1\rangle, |\psi_2\rangle \}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S(\hat{\rho}) = -\sum_n p_n \ln p_n$$

$\hookrightarrow p_n = 0$ initial entropy

measurement: projects $|\psi_1\rangle$ w/ 50%/50% prob. to $|0\rangle$ or $|1\rangle$

$$\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

find ON decomp $\Rightarrow$ diag. basis is $\{|0\rangle, |1\rangle\}$

$$\rho' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow S(\hat{\rho}') = \ln 2$$

P_n

$$> S(\hat{\rho})$$

entropy inc. after measurement

Most general interaction w/ environment:
how does $\hat{\rho}$ transform?
$\Rightarrow$ theory of open (→ inter. w/ env.) quantum systems

Central theorem: Choi-Kraus representation theorem

most general transform looks like:

$$\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \qquad \Gamma \leq N^2$$

N = dim. of Hilbert space

for some operators

$\hat{M}_\gamma \Rightarrow$ Kraus operators (not necessarily unique)

that satisfy: $\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{\mathbb{1}}$ identity

Previous examples:

i) isolated sys: $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger$ $\quad \Gamma = 1$

$\hat{M}_1 = \hat{U}$

$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{U}^\dagger \hat{U} = \hat{\mathbb{1}}$

ii) measurement: $\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger$ $\quad \Gamma = N$

$\hat{M}_\gamma \Rightarrow \hat{P}_a = |a\rangle\langle a|$

$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \sum_a |a\rangle\overbrace{\langle a|a\rangle}^{1}\langle a| = \sum_a |a\rangle\langle a| = \hat{\mathbb{1}}$

Proof: transform a valid $\hat{\rho}$ to another valid dens. op. $\hat{\rho}'$

valid: $\hat{\rho}^\dagger = \hat{\rho}$ $\quad \mathrm{tr}(\hat{\rho}) = 1$ $\quad \langle i|\hat{\rho}|i\rangle \geq 0$ in any basis

also demand that mapping be linear:

example: two ensembles

$\hat{\rho}_A = |\psi_1\rangle\langle\psi_1| \rightsquigarrow \hat{\rho}_A'$

$\hat{\rho}_B = |\psi_2\rangle\langle\psi_2| \rightsquigarrow \hat{\rho}_B'$

combined ensemble (frac. f of A, $1-f$ of B):

$$\hat{\rho} = f\,\hat{\rho}_A + (1-f)\,\hat{\rho}_B \rightsquigarrow \hat{\rho}' = f\,\hat{\rho}_A' + (1-f)\,\hat{\rho}_B'$$