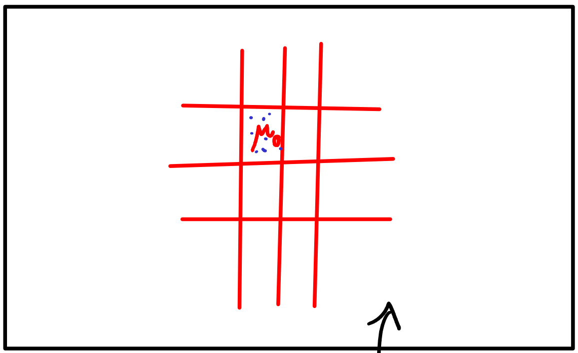


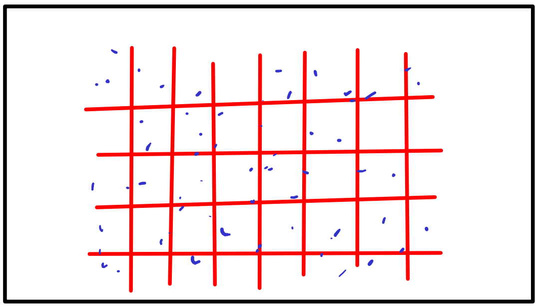
Subspace at const. E

regions: $\Theta(E)$

$P_\mu(t)$ = prob. to be in microstate μ at time t

$$P_\mu(0) = \begin{cases} 1 & \text{if } \mu = \mu_0 \\ 0 & \text{if } \mu \neq \mu_0 \end{cases}$$

mixing as $t \to \infty$



$$P_\mu(t) = \frac{1}{\Theta(E)}$$

microcanonical ensemble

(all energy microstates equally likely)

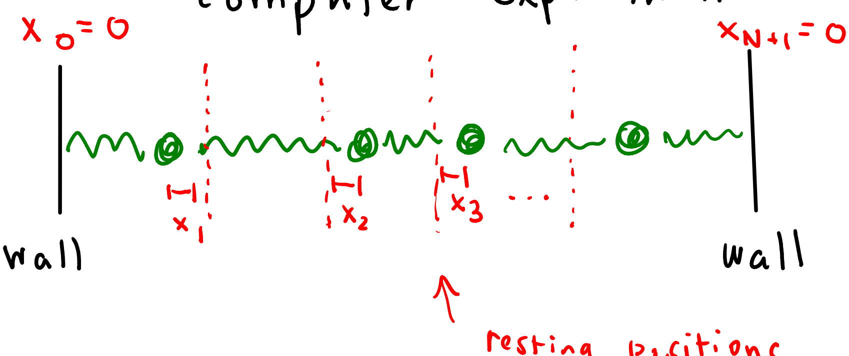
First test: 1953 - 55 Fermi, Pasta, Ulam, Tsingou

(FPUT) first physics computer experiment

connected springs:

N masses $m = 1$

w/ spring const. $k = 1$



eq. of motion:
$n = 1, \dots, N$

$$\ddot{x}_n = \left[(x_{n+1} - x_n) - (x_n - x_{n-1}) \right]$$

Hookean spring forces

$$\cdot \left(1 + \alpha (x_{n+1} - x_{n-1}) \right)$$

↳ nonlinear perturbation when $\alpha \neq 0$

$\underline{\alpha = 0}$

$$\mathcal{H}(\vec{x}, \vec{p}) = \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + \frac{(x_n - x_{n-1})^2}{2} \right) + \frac{x_N^2}{2}$$

$\vec{x} = (x_1, \ldots, x_N)$ $\quad p_n = \dot{x}_n$

$\vec{p} = (p_1, \ldots, p_n)$

Reminder: any funcs $f(\vec{x}, \vec{p})$ $\quad g(\vec{x}, \vec{p})$

$$\{f, g\}_{\vec{x}, \vec{p}} \equiv \sum_{n=1}^{N} \left(\frac{\partial f}{\partial x_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial x_n} \right)$$ Poisson brackets

often left out of notation

time-dep. of any func: $f(\vec{x}, \vec{p})$ $\quad \frac{d}{dt} f = \{f, H\}$

$$f = x_i \Rightarrow \dot{x}_i = \frac{\partial H}{\partial p_i} \qquad f = p_i \Rightarrow \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

$$\{x_i, x_j\} = 0 \qquad \{p_i, p_j\} = 0 \qquad \{x_i, p_j\} = \delta_{ij}$$

canonical transf. (CT) of coordinates:

define $\vec{Q}(\vec{x}, \vec{p})$, $\vec{\mathcal{P}}(\vec{x}, \vec{p})$

+ inverse: $\vec{x}(\vec{Q}, \vec{\mathcal{P}})$ $\quad \vec{p}(\vec{Q}, \vec{\mathcal{P}})$

that preserve Poisson brackets:

$$\{f, g\}_{\vec{x}, \vec{p}} = \{f, g\}_{\vec{Q}, \vec{\mathcal{P}}}$$

$\Rightarrow$ leads:

$$\dot{Q}_n = \{Q_n, H\}_{\vec{Q}, \vec{\mathcal{P}}} = \frac{\partial H}{\partial \mathcal{P}_n}$$

$$\dot{\mathcal{P}}_n = \{\mathcal{P}_n, H\}_{\vec{Q}, \vec{\mathcal{P}}} = -\frac{\partial H}{\partial Q_n}$$

Define a CT for $\alpha = 0$ FPUT system:

$$x_n = \sum_{k=1}^{N} \sqrt{\frac{2I_k}{(N+1)\omega_k}} \sin\left(\frac{nk\pi}{N+1}\right) \sin(\phi_k)$$

$$p_n = \sum_{k=1}^{N} \sqrt{\frac{2I_k\omega_k}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right) \cos(\phi_k)$$

$$\omega_k = 2 \sin\left(\frac{k\pi}{2(N+1)}\right) \qquad k=1,\ldots,N$$

new variables: $\{\phi_1, \ldots, \phi_N, I_1, \ldots\ldots, I_n\}$

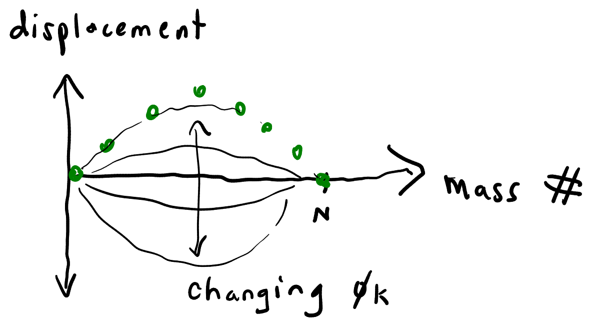
new "position" "momentum"

really: phase of Fourier comp. $\propto$ amplitude of Fourier comp.

check that: $\{x_i, p_j\}_{\vec{\phi}, \vec{I}} = \delta_{ij}$, etc.

after CT $\Rightarrow$ $\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \underbrace{\overset{\text{const.}}{\omega_k} I_k}_{\text{energy of kth mode}}$

k=1

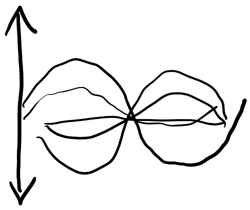


$$\dot{\phi}_i = \frac{\partial \mathcal{H}}{\partial I_i} = \omega_i \Rightarrow \phi_i(t) = \omega_i t + \phi_i(0)$$

$$\dot{I}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i} = 0$$

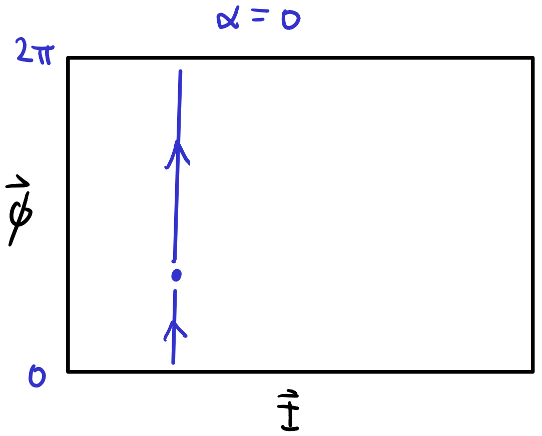
$\Rightarrow$ all I_i are consts. of motion

k=2



when $\alpha \neq 0$:

$$\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \omega_k I_k + \alpha \underbrace{U(\vec{\phi}, \vec{I})}_{\text{complicated func.}}$$



$$\Rightarrow \dot{I}_i = -\frac{\partial H}{\partial \phi_i} \neq 0$$

I_i are no longer consts. of motion