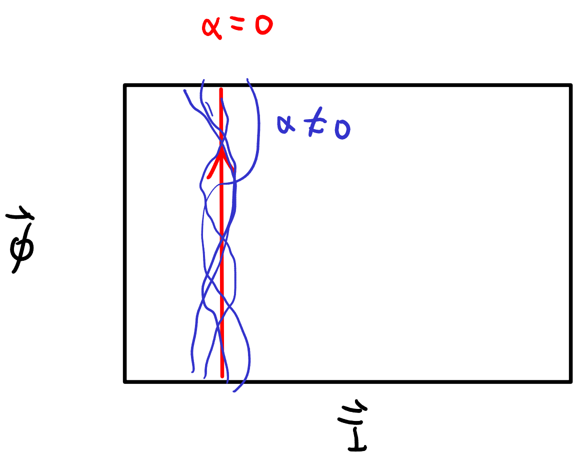
FPUT : n coupled 1D oscillators

normal mode coords: $(\vec{\phi}, \vec{I})$

harmonic springs: $\alpha = 0$ $\quad \mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{n} \omega_k I_k$ ← consts. of motion

non-harmonic springs: $\alpha \neq 0$ $\quad \mathcal{H}(\vec{\phi}, \vec{I}) = \sum \omega_k I_k + \alpha\, U(\phi, I)$

I_k = <u>not</u> consts. of motion



$\alpha \neq 0$: you "quasiperiodic" behavior where you return close but not exactly to initial conditions

$\Rightarrow$ never achieve mixing (microcanonical ensemble) expected by Fermi

Consider a more general class of systems:

$$\mathcal{H}(\vec{x}, \vec{p}) \qquad n \text{ coords} \Rightarrow 2n \text{ dimen. phase space}$$

$\vec{x} \to (x_1, x_2, \ldots, x_n)$ $\quad \vec{p} \to (p_1, \ldots, p_n)$

This system is integrable when:

i) there are n linearly indep. consts. of motion: $F_k(\vec{x}, \vec{p})$ $\quad k = 1, \ldots, n$

$\Rightarrow \{F_k, \mathcal{H}\} = 0$ by convention $F_1 \equiv \mathcal{H}$

ii) $\{F_k, F_\ell\} = 0$ for all k, ℓ

$\Rightarrow$ Liouville-Arnold theorem: for these integrable systems there exists a canonical transf. to "action-angle" coords:

$$(\vec{\phi}, \vec{I})$$

$\vec{\phi}$ = angles $(\phi_1, \ldots, \phi_n)$, $\vec{I}$ = actions $(I_1, \ldots, I_n)$

note: $\phi_i + 2\pi m \equiv \phi_i$ (m integer)

$\Rightarrow$ $\mathcal{H} = \mathcal{H}(I)$ $\quad \dot{I}_k = -\dfrac{\partial \mathcal{H}}{\partial \phi_k} = 0$ $\quad$ all I_k are consts. of motion

$$\dot{\phi}_k = \frac{\partial H}{\partial I_k}(\vec{I}) \equiv \omega_k(\vec{I})$$

$$\phi_k(t) = \omega_k(I)\, t + \phi_k(0)$$

examples of integrable systems:

- all 1D problems w/ conserved energy
- n coupled harmonic springs
- central force problems
- two-body grav. problems
- gyroscopes & tops
- free particles confined on surfaces of ellipsoids

not integrable:
- three body problem (Poincaré)
- dissipative systems
- chaotic systems

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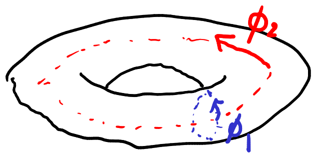
for integrable systems: everything lives on a donut (torus) "hypertorus" $T^n = S^1 \times S^1 \times \dots S^1$

$\uparrow$ circle

$$\vec{\phi}(t) = (\phi_1, \phi_2, \dots, \phi_n)$$ coords on T^n

$n=2$: 

different torus confining motions for every diff. set of $\vec{I}$

$(\vec{\phi}, \vec{I})$ phase space: "foliation" of tori

traj. for integ. system stays on one torus.

classify tori:

i) resonant tori: there exists a vector $\vec{v}$ of integers, $\vec{v} \neq 0$ and $v_i \in \mathbb{Z}$ such that $\vec{v} \cdot \vec{\omega} = 0$

$$\vec{\omega} = (\omega_1(\vec{I}), \omega_2(\vec{I}), \dots, \omega_n(\vec{I}))$$

Special case: $\omega_i = z_i \omega^*$ ↳ const. ↳ integer

$\Rightarrow$ periodic orbits on torus

ii) non-resonant tori: no such $\vec{v}$ exists

traj. densely fills torus (quasiperiodicity)

What happens if you break integrability via a small perturbation: $\mathcal{H}(\vec{\phi}, \vec{I}) = \mathcal{H}_0(\vec{I}) + \alpha \mathcal{H}_1(\vec{\phi}, \vec{I})$

integrable $\quad \alpha$ small

Kolmogorov, Arnold, Moser (KAM) theorem: 1954-63

- focus on "non-degenerate" systems:

$$\frac{\partial \omega_i}{\partial I_j} = \frac{\partial^2 H_0}{\partial I_i \partial I_j} \equiv M_{ij} \qquad M = n \times n \text{ matrix}$$

$\det M \neq 0$ (FPUT is excluded!)

only proven for FPUT at low energies: Rink arxiv: 0506024

- focus on tori which are "strongly" non-resonant (SNR):

$$\vec{v} \cdot \vec{\omega} \geq \frac{\epsilon}{|\vec{v}|^{\tau}} \quad \text{for some} \quad \left.\begin{matrix} \tau > n-1 \\ \epsilon > 0 \end{matrix}\right\} \in \mathbb{R}$$

$$\text{for any } \vec{v} \in \mathbb{Z}^n$$

note: for small ϵ, most tori are SNR

- KAM theorem: there exists $\delta > 0$ such that for small perturb. $\alpha \leq \delta \epsilon^2$ all SNR tori survive, being only slightly deformed $\Rightarrow$ traj. are stuck on "deformed" donuts