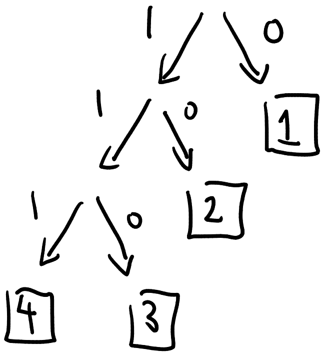
$N = 4$ states: Bob: 2 4 1 3 1 2 . . . . .

each state occurs w/ some prob $p_n(t)$

encode: 10 111 0 11 0 0

every code (prefix-code) expressed in terms of binary tree:



state	code	ℓ_n
1	0	1
2	10	2
3	110	3
4	111	3

avg. # bits / state

$$B = \sum_n p_n(t)\, \ell_n$$

Shannon's source coding thm.

Today: prove $B \geq B_{min} = -\sum_n p_n \log_2 p_n = \frac{S}{k_B \ln 2}$

Consider any code (any binary tree):

fake probability $q_n \equiv 2^{-\ell_n} = \left(\frac{1}{2}\right)^{\ell_n}$

= prob. of reaching state n (leaf of a tree) if you start at tree top + choose each fork w/ equal prob.

$$\sum_n q_n = 1$$

$$B = \sum_n p_n \ell_n = -\sum_n p_n \log_2 q_n$$

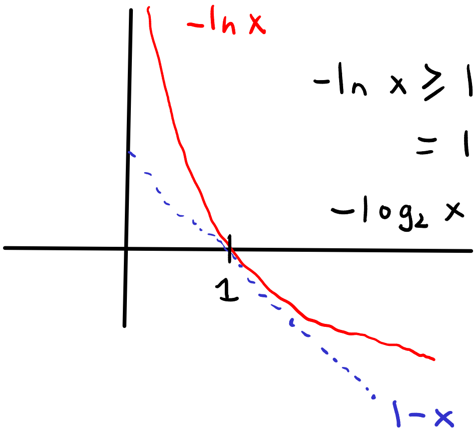
Shannon's claim $\quad B - B_{min} \geq 0$

$$-\sum_n p_n \log_2 q_n - \left(-\sum_n p_n \log_2 p_n\right)$$

$$= -\sum_n p_n \log_2 \frac{q_n}{p_n} \geq \frac{1}{\ln 2} \sum_n p_n \left(1 - \frac{q_n}{p_n}\right)$$

$$= \frac{1}{\ln 2}\left(\underbrace{\sum_n p_n}_{1} - \underbrace{\sum_n q_n}_{1}\right)$$

$$= 0$$



$-\ln x \geq 1-x$

$= 1-x \quad$ iff $x=1$

$-\log_2 x = \dfrac{-\ln x}{\ln 2}$

$\Rightarrow$ B_{min} is a universal lower bound!

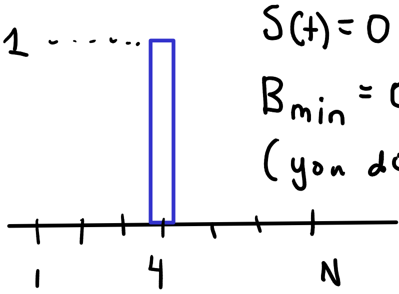
$B = B_{min}$ iff $q_n = p_n$ for each n ($x=1$ for each term)

$2^{-l_n} = p_n$

$\Rightarrow$ $l_n = -\log_2 p_n$ in an optimal code

note: $-\log_2 p_n$ is not always integer so in general we can approach but not reach B_{min}

Summary: how much info about the state of sys. is contained in prob. dist $\vec{p}(t)$?



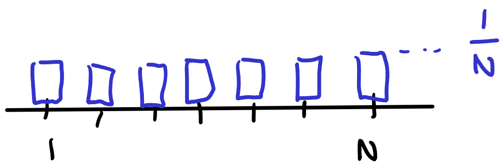
$S(t) = 0$

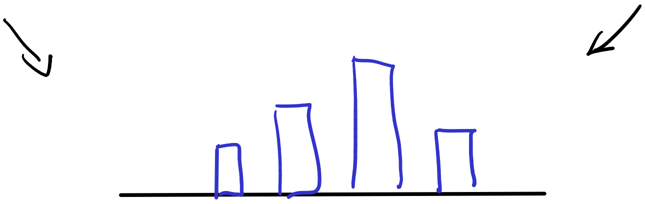
$B_{min} = 0$ bits

(you do not need to send message about state)

$S(t) = k_B \ln N$

$B_{min} = \log_2 N$ bits





$0 < S(t) < k_B \ln N$

$0 < B_{min} < \log_2 N$

Return to physics:

$\Delta F = \Delta E - T\Delta S$

1st : $Q = \Delta E + W$

2nd: $I = -\dfrac{\Delta F}{T} + \dfrac{W}{T} \geq 0$

special cases:

1) $Q = W = 0 \Rightarrow \Delta E = 0$

$\Delta F = -T\Delta S = -TI \leq 0$

$\Rightarrow \Delta S \geq 0$ (entropy max.)

2) $Q \neq 0, W = 0$

$\Delta F = -TI \leq 0$ (free energy min.)

3) $Q \neq 0, W \neq 0$

$\Rightarrow W = -\Delta F - TI \qquad I \geq 0$

$\leq -\Delta F$

if you want $\underbrace{W > 0}_{\text{sys. doing work}} \Rightarrow \Delta F < 0$

but also know $F(t) = E(t) - TS(t) \geq F_{min}$

(bounded from below)

while $F(t)$ is decreasing $\Rightarrow W > 0$

but eventually we run out (reach F_{min})

$\Rightarrow$ hence $F(t)$ is "free energy" $\Rightarrow$

energy available to do work ("gas in the tank")

What about stationary state when $t \to \infty$?

if $p_n(t) \to p_n^s$: $E(t) = \sum_n p_n(t) E_n \to E^s$

$S(t) = -k_B \sum_n p_n(t) \ln p_n(t) \to S^s$

if t_0 + t_τ in stat. state:

$$\Delta E = E(t_\tau) - E(t_0) = 0$$

$$\Delta S = S(t_\tau) - S(t_0) = 0$$

$$\Rightarrow \Delta F = 0$$

1st: $Q = \cancel{\Delta E}^{\,0} + W \Rightarrow Q = W$

2nd: $I = -\frac{\cancel{\Delta F}^{\,0}}{T} - \frac{W}{T} \Rightarrow I = -\frac{W}{T} \geq 0$

Stationary state (sys. w/ env. at temp. T) as $t \to \infty$: $Q = W = -TI \leq 0$

two types: 1) $Q = W = 0$ $(I = 0)$

equil. stat. state (ESS)

$$p_n^s = \frac{e^{-\beta E_n}}{Z} \quad \text{if } E_n \text{ are diff.}$$

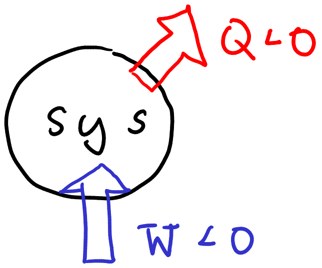
$$= \frac{1}{N} \quad \text{if all } E_n \text{ are const.}$$

2) $Q = W = -TI < 0$

noneq. stat. state (NESS)

to maintain: $W < 0$: do net work <u>on</u> sys

$Q < 0$: system dumps heat into env.



$$W = W_{out} - W_{in} < 0$$

W_{out}: work done by sys > 0

W_{in}: work done on sys. < 0

perpetual motion:

impossible