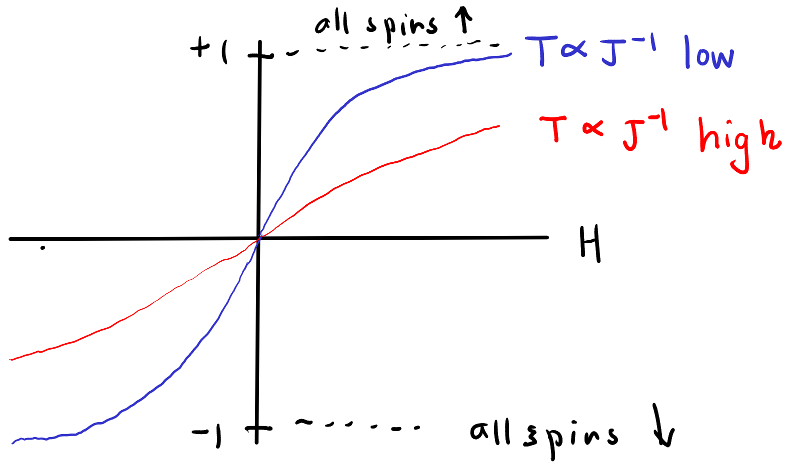
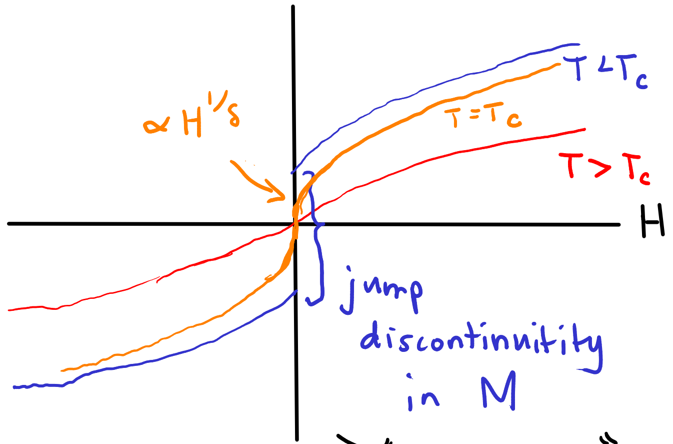
Ising; $N=2$ •—•

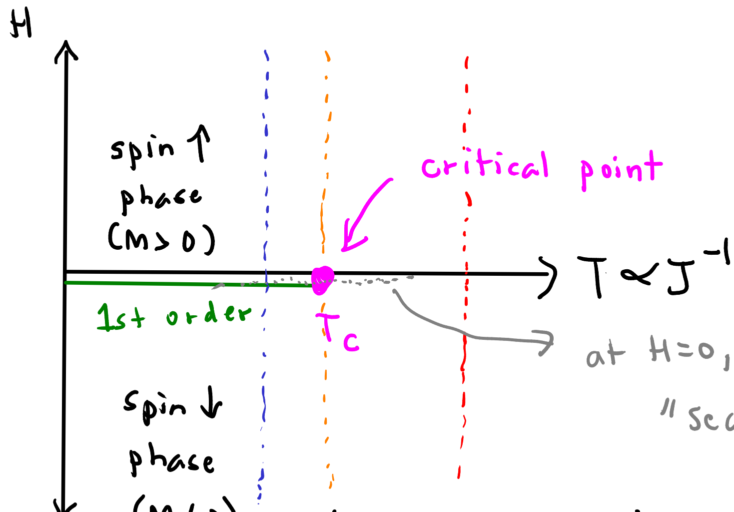
2D square lattice $N \to \infty$ [Onsager, 1944, $H=0$]

$$M(J,H) = \left\langle \frac{1}{N} \sum_i s_i \right\rangle \propto \frac{\partial}{\partial H} \ln Z$$





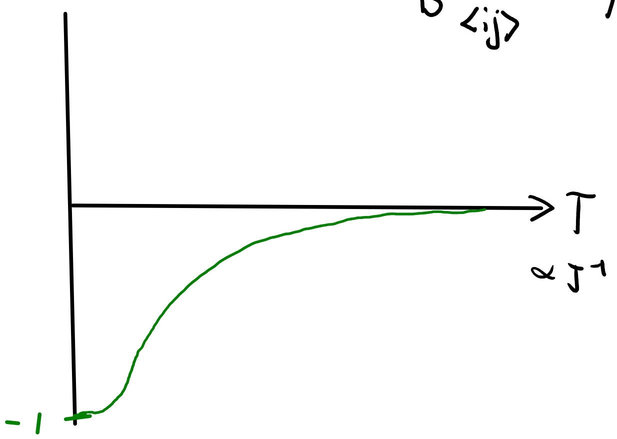
⇒ "first order" phase trans. b/c discont. in 1st deriv. of $\ln Z$

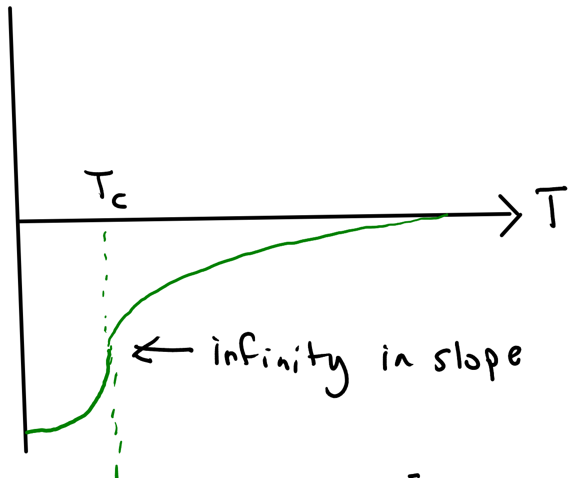


at $H=0$, move thru $T=T_c$ ⇒ "second order" or continuous phase trans. b/c no discont. in M or U

$\underbrace{\qquad\qquad}_{\text{ferromag. phase } (T<T_c)}$ $\underbrace{\qquad\qquad}_{\text{paramagnetic phase } (T>T_c)}$

$$U(J,H=0) = -\left\langle \frac{1}{B} \sum_{\langle ij \rangle} s_i s_j \right\rangle \propto \frac{\partial}{\partial J} \ln Z$$





$$C = \left.\frac{dU}{dT}\right|_{H=0} = -\frac{k_B J^2}{j} \left.\frac{\partial}{\partial J} U\right|_{H=0}$$

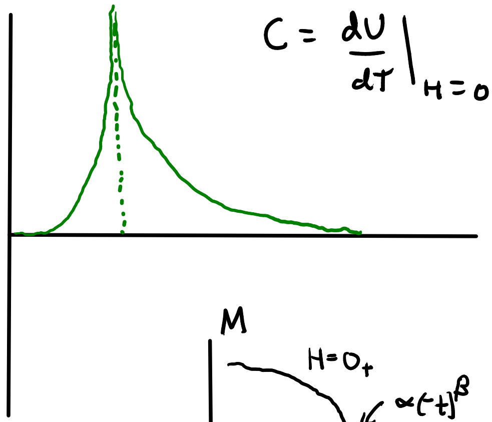
$$= +\frac{k_B J^2}{jB} \left.\frac{\partial^2}{\partial J^2} \ln Z\right|_{H=0}$$

specific heat per bond at $H=0$

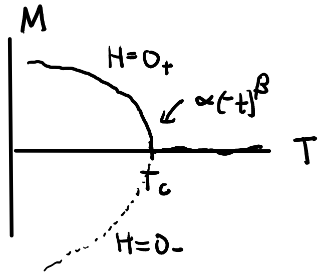
$$T = \frac{j}{k_B J}$$

$$\Rightarrow \frac{d}{dT} = -\frac{k_B J^2}{j} \frac{d}{dJ}$$

Singularities at critical point characterized by critical exponents:



$C = \frac{dU}{dT}\Big|_{H=0}$



at $T=T_c$, $H\to 0^+$: $M \propto H^{1/\delta}$

$H \to 0^-$: $M \propto -H^{1/\delta}$

at $H = 0_\pm$, $T<T_c$: $M \propto (-t)^\beta$

$t = \frac{T-T_c}{T_c}$ for small t

$t=0$ at $T=T_c$

spontaneous symm. breaking: M is "order param" discont $\propto 2(-t)^\beta$

at $H=0$, $T<T_c$: $C \propto (-t)^{-\alpha'}$

$T>T_c$ $\quad C \propto t^{-\alpha}$

2D Ising: $\delta = 15$

$\beta = 1/8$

$\alpha = \alpha' = 0$

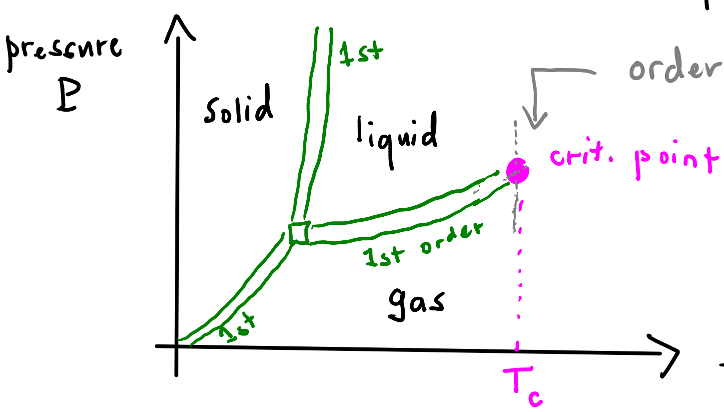
↳ technically in 2D Ising not a power law for C, but $C \propto |\ln t|$

3D Ising: $\delta = 4.78\ldots$

$\beta = .326\ldots$

$\alpha = \alpha' = .11\ldots$

typical solid-liquid-gas phase diagram:



order parameter: $\rho - \rho_c$

density

density at crit. point

analogue of H: $P - P_c$

pressure at crit.

$$(\rho - \rho_c) = \pm (P - P_c)^{1/\delta}$$

Observation: many liquid-gas crit. points (indep. of molecular details) exponents are same as 3D Ising model.

Universality: often a broad range of Hamiltonians that have same exponents at crit. points (though temps, molec. details, etc. are diff.)

Another: add next-nearest neighbor ferromag interactions to 2D Ising model $\Rightarrow$ same crit. exponents

Two big questions:

- why the singularities?
- why universality?