Renormalization group theory (RG)

early beginnings : Leo Kadanoff (1966)

Ken Wilson (1971)

My Life w/ Fisher $\longleftarrow$ Michael Fisher + $\underbrace{\text{Wilson}}$ (1972)

arXiV : N. David Mermin

Nobel Prize (1982)

RG in a nutshell:

orig. Hamilt. $\mathcal{H}_{\vec{s}}(\vec{K})$ $\rightsquigarrow$ RG mapping "coarse-grained" $\mathcal{H}_{\vec{s}'}(\vec{K}')$

N sites ("degrees of freedom") where $N' < N$

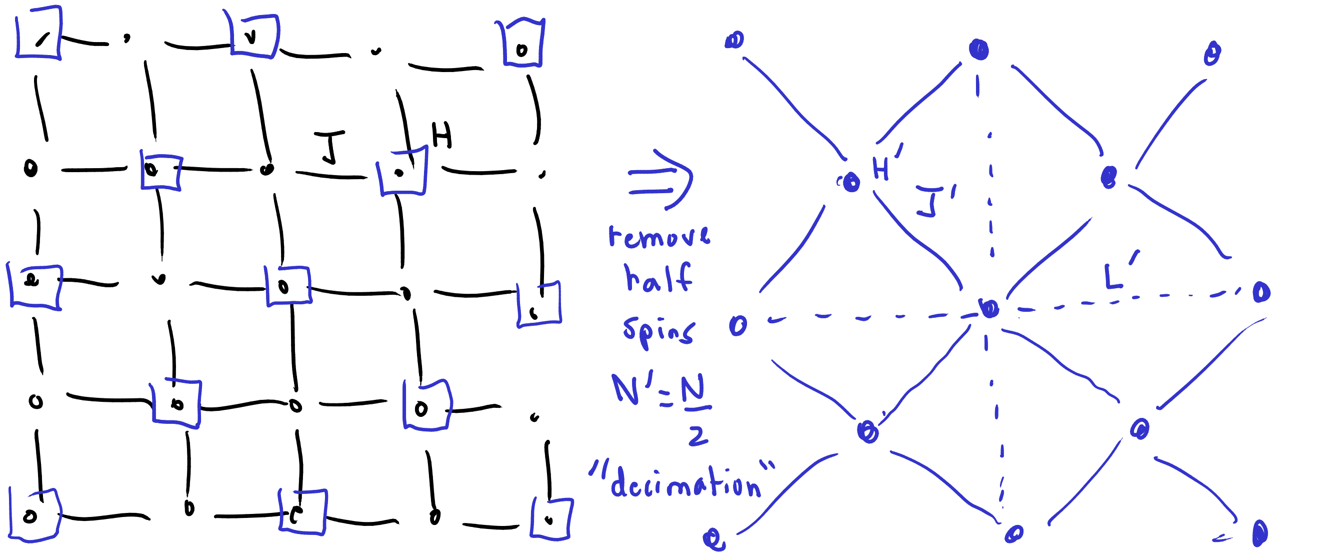
such that partition func. Z is conserved:

$$Z = \sum_{\vec{s}} e^{-\beta \mathcal{H}_{\vec{s}}(\vec{K})} = \sum_{\vec{s}'} e^{-\beta \mathcal{H}_{\vec{s}'}(\vec{K}')}$$

$$\vec{K} = (J, H, \underbrace{0, 0, 0, \ldots}_{\text{all other terms in } \mathcal{H} \text{ are zero for Ising model}})$$

$\Big\downarrow$ RG

$$\vec{K}' = (J', H', \underbrace{\ldots}_{\text{other terms may be nonzero}})$$



old lattice
J, H

new lattice
J', H', L' (n.n.n. interactions) etc.

$$Z = \sum_{\vec{s}} e^{-\beta \mathcal{H}_{\vec{s}}(\vec{K})} = \sum_{\vec{s}'} \underbrace{\sum_{\vec{s}_{dec}} e^{-\beta \mathcal{H}_{\vec{s}}(\vec{K})}} \equiv \sum_{\vec{s}'} \underbrace{e^{-\beta \mathcal{H}_{\vec{s}'}(\vec{K}')}}$$

$\vec{s}'$: spins in new lattice

$\vec{s}_{dec}$: spins to decimate

$$\Rightarrow \quad e^{-\beta \mathcal{H}_{\vec{s}'}(\vec{K}')} = \sum_{\vec{s}_{dec}} e^{-\beta \mathcal{H}_{\vec{s}}(\vec{K})}$$

must be true for all possible $\vec{s}'$

you may need addit. terms in $\mathcal{H}_{\vec{s}'}(\vec{K}')$ to make this work

$\Rightarrow$ multiple RG steps: $\vec{K} \rightarrow \vec{K}' \rightarrow \vec{K}'' \rightarrow \dots$

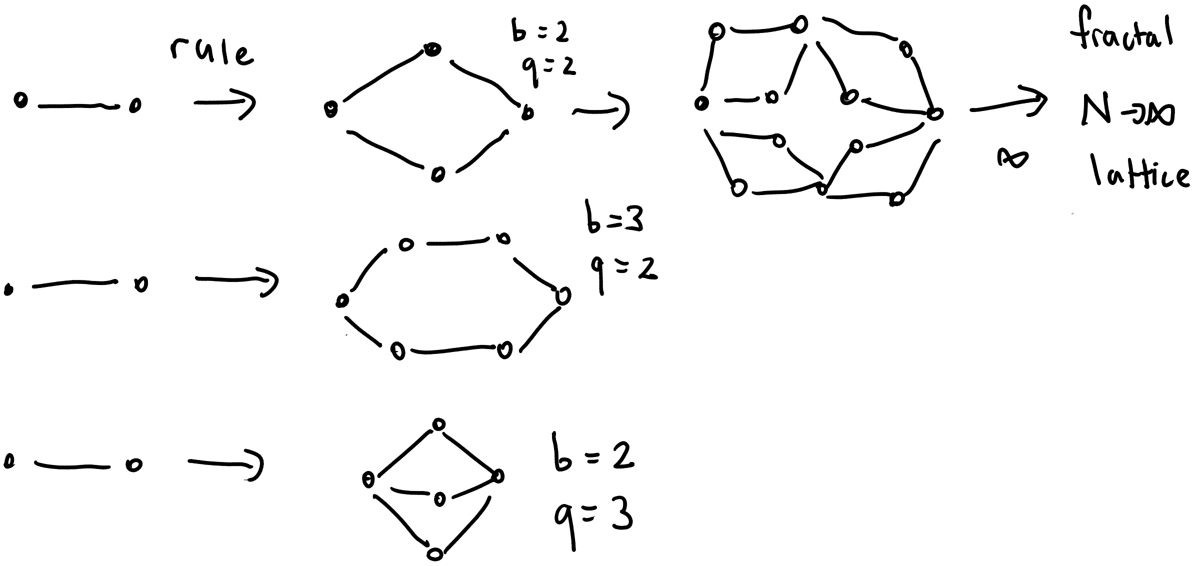
"flow" in an ∞-dim. Hamilt. param space

For regular lattices (i.e. square) RG can only be done approximately (i.e. ignore addit. terms).

But certain fractal lattices it can be done exactly: Berker & Ostlund J. Phys. C 1979

"hierarchical lattices"

example:



parameters: $b \Rightarrow$ each bond replaced by b bonds in series

$q \Rightarrow$ # parallel sets

geometry: B_n = # bonds after nth const. step.

$$= (qb)^n$$

sites added in nth step:

$$= q(b-1)\, B_{n-1}$$

N_n = # sites after n steps:

$$= 2 + \sum_{k=1}^{n} q(b-1)\, B_{n-1}$$

$$= 2 + (b-1)\, q\, \left((qb)^n - 1\right) (qb-1)^{-1}$$

$$\xrightarrow[n\to\infty]{} \frac{q(b-1)}{qb-1}(qb)^n \propto B^n$$

dimensionality: $N \propto \ell^{d}$

$d \to$ dimension

N ↑ # sites in lattice

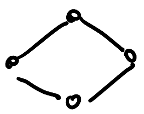
$\ell \to$ diameter of lattice

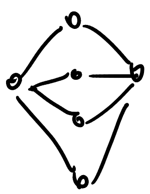
diameter : $\ell_n = b^n$

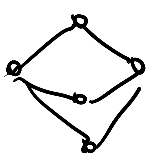
(shortest dist. b/t orig. 2 sites)

$$
\begin{aligned}
N_n &\propto (qb)^n \\
&= q^n b^n \\
&= b^{n + n\log_b q} \\
&= b^{n\left(1+\frac{\log q}{\log b}\right)} \\
&= \ell_n^{\,1+\frac{\log q}{\log b}}
\end{aligned}
$$

$\Rightarrow$ dim. $d = 1 + \dfrac{\log q}{\log b}$

 $b=2,\ q=2 \Rightarrow d=2$

 $b=2,\ q=4 \Rightarrow d=3$

 $b=2,\ q=3 \Rightarrow d = 2.585\ldots$

RG on a $d=2$ fractal lattice (focus on $H=0$ subspace)

for comparison: $d=2$ square lattice