

closed sys:  $\hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t)$

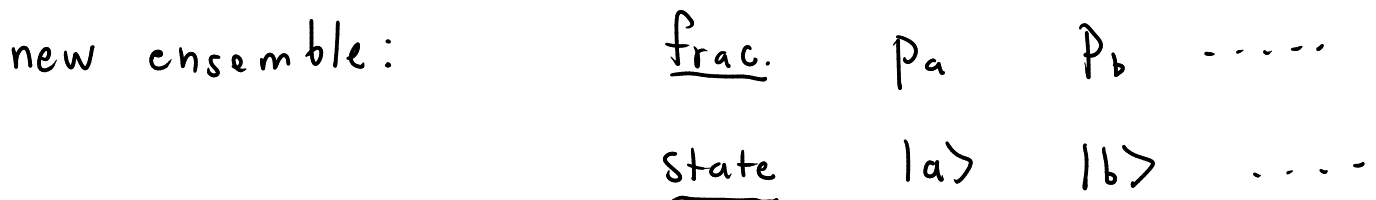
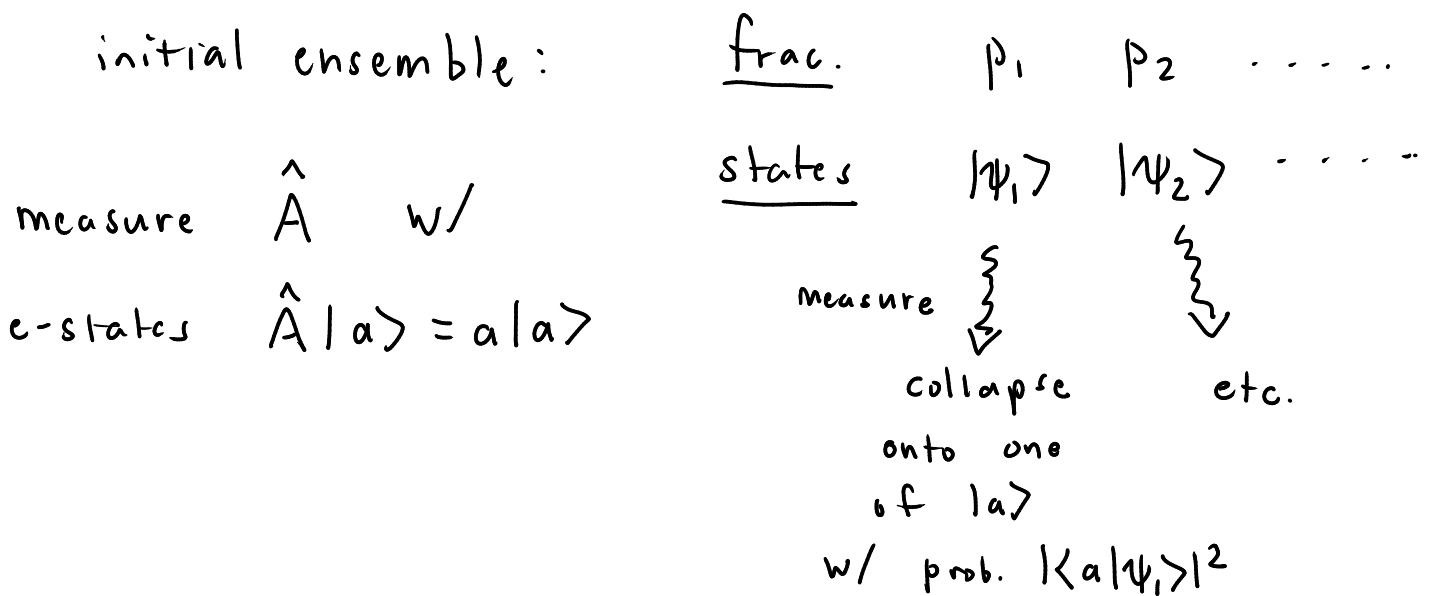
$\Rightarrow S(\hat{\rho}(t)) = S(\hat{\rho}(0))$

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What about an open system, interacting w/ some outside (macroscopic) env.

Start w/ one scenario we already understand: measurements.

ensemble under measurement:



$$p_a = \sum_n p_n |\langle a|\psi_n\rangle|^2$$

↑  
frac. of  $|\psi_n\rangle$  in orig. ensemble

new density operator:  $\hat{\rho}' = \sum_a p_a |a\rangle\langle a|$

$$\hat{P}_a = |a\rangle\langle a|$$

= projection operator onto  $|a\rangle$

$$\hat{P}_a^\dagger = |a\rangle\langle a| = \hat{P}_a$$

$$= \sum_{n,n} p_n \langle a|\psi_n\rangle \langle \psi_n|a\rangle |a\rangle\langle a|$$

$$= \sum_a \underbrace{|a\rangle\langle a|}_{\hat{P}_a} \left[ \sum_n p_n |\psi_n\rangle\langle \psi_n| \right] \underbrace{|a\rangle\langle a|}_{\hat{P}_a}$$

orig. density operator

$$\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger$$

how dens. op. changes under meas. of  $\hat{A}$

note:  $\sum_a \hat{P}_a^\dagger \hat{P}_a = \hat{I}$

$$\sum_a |a\rangle\langle a|a\rangle\langle a| = \sum_a |a\rangle\langle a|$$

example: qubit,  $|0\rangle + |1\rangle$

initially prepare 100% of ensemble in state  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

init. dens. oper.  $\hat{\rho} = |\psi_1\rangle\langle \psi_1|$

calc. entropy: find basis where  $\hat{\rho}$  is diag.

$$\{|\psi_1\rangle, |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$$

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\langle \psi_1 | \hat{\rho} | \psi_1 \rangle = 1$$

..... = 0

↳ e-vals  $p_n$

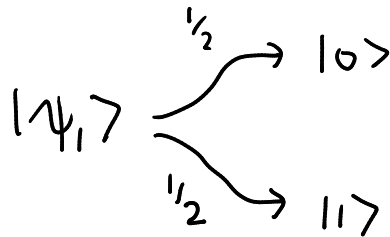
$$S(\hat{\rho}) = - \sum_n p_n \ln p_n = 1 \ln 1 + 0 \ln 0 = 0$$

initial entropy

measure  $\hat{H} \Rightarrow$  collapse  $|\psi_1\rangle$  onto  $|0\rangle$  or  $|1\rangle$

$$\hat{H}|0\rangle = E_0|0\rangle$$

$$\hat{H}|1\rangle = E_1|1\rangle$$



$$|\langle\psi_1|0\rangle|^2 = \frac{1}{2}$$

$$\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

entropy: diagonalize  $\Rightarrow$  basis  $\{|0\rangle, |1\rangle\}$

$$\hat{\rho}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow S(\hat{\rho}') = \frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2}$$

$$= \ln 2$$

$$> S(\hat{\rho})$$

entropy increases after measurement!

two ways so far to change  $\hat{\rho}$ :

i) closed:  $\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^\dagger \quad \hat{U}^\dagger \hat{U} = \hat{I}$

ii) measurement:  $\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger \quad \sum_a \hat{P}_a^\dagger \hat{P}_a = \hat{I}$

transformations map valid dens. oper.  $\hat{\rho}$   
 onto new valid dens. oper.  $\hat{\rho}'$

$$\hat{\rho}^+ = \hat{\rho}$$

$$\text{tr}(\hat{\rho}) = 1$$

$\underbrace{\langle i | \hat{\rho} | i \rangle}_{\text{prob. to observe } |i\rangle \text{ in ensemble}} \geq 0$  in any basis

$$\hat{\rho}'^+ = \hat{\rho}'$$

$$\text{tr}(\hat{\rho}') = 1$$

$\langle i | \hat{\rho}' | i \rangle \geq 0$  in any basis



these transforms are also linear:

two ensembles:  $\hat{\rho}_A = |\psi_1\rangle\langle\psi_1| \xrightarrow{\text{time}} \hat{\rho}'_A$   
 (pure states)  $\hat{\rho}_B = |\psi_2\rangle\langle\psi_2| \xrightarrow{\text{time}} \hat{\rho}'_B$

combined ensemble: take frac.  $f$  of A and  $1-f$  of B:

$$\begin{aligned} \hat{\rho} &= f |\psi_1\rangle\langle\psi_1| + (1-f) |\psi_2\rangle\langle\psi_2| \\ &= f \hat{\rho}_A + (1-f) \hat{\rho}_B \end{aligned}$$

$$\rightsquigarrow \hat{\rho}' = f \hat{\rho}'_A + (1-f) \hat{\rho}'_B$$

theorem: Choi-Kraus representation theorem

most general transf. of  $\hat{\rho}$  that satisfies all above properties

looks like:

$$\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_{\gamma} \hat{\rho} \hat{M}_{\gamma}^{\dagger} \quad \text{where} \quad \sum_{\gamma} \hat{M}_{\gamma}^{\dagger} \hat{M}_{\gamma} = \hat{I}$$

$\Gamma \leq N^2$        $N = \dim.$  of Hilbert space  
 $N=2$  qubit:  $\Gamma \leq 4$

previous examples:

i) closed:  $\Gamma = 1$   
 $\hat{M}_1 = \hat{U}$   
 $\sum_{\gamma} \hat{M}_{\gamma}^{\dagger} \hat{M}_{\gamma} = \hat{U}^{\dagger} \hat{U} = \hat{I}$

ii) measurement:  $\Gamma = N$        $\hat{P}_a^{\dagger} = \hat{P}_a$   
 $\hat{M}_{\gamma} \Rightarrow \hat{P}_a = |a\rangle\langle a|$   
 $\sum_{\gamma} \hat{M}_{\gamma}^{\dagger} \hat{M}_{\gamma} = \sum_a |a\rangle\langle a|a\rangle\langle a| = \hat{I}$

in general:  $\hat{M}_{\gamma}^{\dagger} \neq \hat{M}_{\gamma}$

note:  $\hat{M}_{\gamma}$  not necessarily unitary

$\sum_{\gamma} \hat{M}_{\gamma}^{\dagger} \hat{M}_{\gamma} \neq \hat{I}$  in general

$\hat{M}_{\gamma} \propto |1\rangle\langle 0|$  example  
 $= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

note: set of  $\{\hat{M}_\alpha\}$  matrices is not  
unique  $\Rightarrow$  we can find new sets  
via linear combos of old ones