closed sys: $\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^\dagger(t)$

$$\Rightarrow \quad S(\hat{\rho}(t)) = S(\hat{\rho}(0))$$

What about an <u>open</u> system, interacting w/ some outside (macroscopic) env.

Start w/ one scenario we already understand: measurements.

ensemble under measurement:

initial ensemble:

<u>frac.</u>	p_1	p_2	.....
<u>states</u>	$\lvert\psi_1\rangle$	$\lvert\psi_2\rangle$	.....

measure $\hat{A}$ w/ e-states $\hat{A}\lvert a\rangle = a\lvert a\rangle$

measure → collapse onto one of $\lvert a\rangle$ w/ prob. $\lvert\langle a\vert\psi_1\rangle\rvert^2$; etc.

new ensemble:

<u>frac.</u>	p_a	p_b	.....
<u>state</u>	$\lvert a\rangle$	$\lvert b\rangle$	....

$$p_a = \sum_n p_n \lvert\langle a\vert\psi_n\rangle\rvert^2$$

↑ frac. of $\lvert\psi_n\rangle$ in orig. ensemble (p_n)

new density operator: $\hat{\rho}' = \sum_a p_a \lvert a\rangle\langle a\rvert$

$$= \sum_{a,n} p_n \langle a|\psi_n\rangle\langle\psi_n|a\rangle\, |a\rangle\langle a|$$

$\hat{P}_a = |a\rangle\langle a|$

= projection operator onto $|a\rangle$

$\hat{P}_a^\dagger = |a\rangle\langle a| = \hat{P}_a$

$$= \sum_a \underbrace{|a\rangle\langle a|}_{\hat{P}_a} \underbrace{\left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right]}_{\text{orig. density operator}} \underbrace{|a\rangle\langle a|}_{\hat{P}_a}$$

$$\boxed{\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger}$$

how dens. op. changes under meas. of $\hat{A}$

note: $\sum_a \hat{P}_a^\dagger \hat{P}_a = \hat{I}$

$$\sum_a \hat{P}_a^\dagger \hat{P}_a = \sum_a |a\rangle\langle a|a\rangle\langle a| = \sum_a |a\rangle\langle a|$$

example: qubit, $|0\rangle$ + $|1\rangle$

initially prepare 100% of ensemble in state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

init. dens. oper. $\hat{\rho} = |\psi_1\rangle\langle\psi_1|$

calc. entropy: find basis where $\hat{\rho}$ is diag.

$$\left\{ |\psi_1\rangle, |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$$

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\langle\psi_1|\hat{\rho}|\psi_1\rangle = 1$

$\ldots\ldots = 0$

↳ e-vals p_n

$$S(\hat{\rho}) = -\sum_n p_n \ln p_n = 1 \ln 1 + 0 \ln 0 = 0$$

initial entropy

measure $\hat{H} \Rightarrow$ collapse $|\psi_1\rangle$ onto $|0\rangle$ or $|1\rangle$

$$\hat{H}|0\rangle = E_0|0\rangle$$
$$\hat{H}|1\rangle = E_1|1\rangle$$

$$|\psi_1\rangle \xrightarrow{1/2} |0\rangle \qquad |\psi_1\rangle \xrightarrow{1/2} |1\rangle$$

$$|\langle\psi_1|0\rangle|^2 = \frac{1}{2}$$

$$\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

entropy: diagonalize $\Rightarrow$ basis $\{|0\rangle, |1\rangle\}$

$$\hat{\rho}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow S(\hat{\rho}') = \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{1}{2}$$
$$= \ln 2$$
$$> S(\hat{\rho})$$

entropy increases after measurement!

two ways so far to change $\hat{\rho}$:

i) closed; $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger \qquad \hat{U}^\dagger\hat{U} = \hat{I}$

ii) measurement: $\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger \qquad \sum_a \hat{P}_a^\dagger \hat{P}_a = \hat{I}$

transformations map valid dens. oper. $\hat{\rho}$ onto new valid dens. oper. $\hat{\rho}'$

$\hat{\rho}^\dagger = \hat{\rho}$ $\qquad\qquad$ $\hat{\rho}'^\dagger = \hat{\rho}'$

$\mathrm{tr}(\hat{\rho}) = 1$ $\quad \rightsquigarrow \quad$ $\mathrm{tr}(\hat{\rho}') = 1$

$\underbrace{\langle i|\hat{\rho}|i\rangle}_{\text{prob. to observe } |i\rangle \text{ in ensemble}} \geq 0$ in any basis $\qquad$ $\langle i|\hat{\rho}'|i\rangle \geq 0$ in any basis

these transforms are also linear:

two ensembles: (pure states)

$$\hat{\rho}_A = |\psi_1\rangle\langle\psi_1| \overset{\text{time}}{\rightsquigarrow} \hat{\rho}_A'$$

$$\hat{\rho}_B = |\psi_2\rangle\langle\psi_2| \rightsquigarrow \hat{\rho}_B'$$

combined ensemble: take frac. f of A and 1-f of B:

$$\hat{\rho} = f|\psi_1\rangle\langle\psi_1| + (1-f)|\psi_2\rangle\langle\psi_2|$$

$$= f\hat{\rho}_A + (1-f)\hat{\rho}_B$$

$$\rightsquigarrow \quad \hat{\rho}' = f\hat{\rho}_A' + (1-f)\hat{\rho}_B'$$

theorem: Choi-Kraus representation theorem

most general transf. of $\hat{\rho}$ that satisfies all above properties looks like:

$$\boxed{\underbrace{\hat{\rho}'}_{N\times N} = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \underbrace{\hat{\rho}}_{N\times N} \underbrace{\hat{M}_\gamma^+}_{N\times N} \quad \text{where} \quad \sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \hat{I}}$$

$\Gamma \leq N^2$ $\quad$ $N =$ dim. of Hilbert space

$N=2$ qubit : $\Gamma \leq 4$

previous examples:

i) closed: $\Gamma = 1$

$\hat{M}_1 = \hat{U}$

$$\sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \hat{U}^+ \hat{U} = \hat{I}$$

ii) measurement : $\Gamma = N$ $\qquad$ $\hat{P}_a^+ = \hat{P}_a$

$\hat{M}_\gamma \Rightarrow \hat{P}_a = |a\rangle\langle a|$

$$\sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \sum_a |a\rangle\langle a|a\rangle\langle a| = \hat{I}$$

in general: $\hat{M}_\gamma^+ \neq \hat{M}_\gamma$

$\hat{M}_\gamma$ not necessarily unitary

$\hat{M}_\gamma \propto |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ example

note:
$\sum_\gamma \hat{M}_\gamma \hat{M}_\gamma^+ \neq \hat{I}$ in general

note: set of $\{\hat{M}_\gamma\}$ matrices in not unique $\Rightarrow$ we can find new sets via linear combo's of old ones