Choi-Kraus: (CK) open sys $\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^{\dagger}$ ← Kraus matrices

$$\Gamma \le N^2$$

$$\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{I}$$

Goal: quantum master equ

$$\frac{\partial}{\partial t} \hat{\rho}(t) = \sim\sim\sim$$

$$\hat{\rho}(t+\delta t) = \sum_\gamma \hat{M}_\gamma \hat{\rho}(t) \hat{M}_\gamma^\dagger$$

implicitly: $\hat{\rho}(t+\delta t)$ depends only on $\hat{\rho}(t)$ [Markovian]

$$\hat{\rho}(t+\delta t) = \hat{\rho}(t) + \sim\sim \delta t$$

special case: no interactions w/ env.

time-ind. sys. Hamilt.

$$\hat{\rho}(t+\delta t) = \hat{U}_s \hat{\rho}(t) \hat{U}_s^\dagger \qquad \hat{U}_s = e^{-i\hat{H}_s \delta t/\hbar}$$

$$\hat{M}_1 = \hat{U}_s \qquad \hat{M}_\gamma = 0 \qquad \gamma > 1$$

$$\hat{\rho}(t+\delta t) = \left(\hat{I} - \frac{i\hat{H}_s \delta t}{\hbar} + \dots\right) \hat{\rho}(t) \cdot \left(\hat{I} + \frac{i\hat{H}_s \delta t}{\hbar} + \dots\right)$$

$$= \hat{\rho}(t) - \frac{i}{\hbar}[\hat{H}_s, \hat{\rho}(t)]\delta t + \dots$$

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{\mathcal{H}}_s, \hat{\rho}]}$$

qu. master equ for isolated sys (von Neumann equ.)

turn on interactions:

$$\hat{M}_1 \approx \hat{I} - \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \underbrace{\hat{K}\,\delta t}_{\text{correc. due to env.}}$$

$\gamma > 1$: $\quad \hat{M}_\gamma = \sqrt{\delta t}\,\hat{L}_\gamma \quad$ for some operators $\hat{L}_\gamma$

no env: $\quad \hat{L}_\gamma, \hat{K} \to 0$

demand:
$$\hat{I} = \sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma$$
$$= \hat{I} + \delta t \underbrace{\left[2\hat{K} + \sum_{\gamma>1} \hat{L}_\gamma^\dagger \hat{L}_\gamma\right]}_{=0} + \cdots$$

$$\Rightarrow \quad \hat{K} = -\frac{1}{2}\sum_{\gamma>1} \hat{L}_\gamma^\dagger \hat{L}_\gamma$$

plug into CK theorem:

algebra :

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{\mathcal{H}}_s, \hat{\rho}] + \sum_{\gamma>1}\left(\hat{L}_\gamma \hat{\rho} \hat{L}_\gamma^\dagger - \frac{1}{2}\hat{L}_\gamma^\dagger \hat{L}_\gamma \hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_\gamma^\dagger \hat{L}_\gamma\right)}$$

quantum master equ. (Lindblad eq.) 1976

Goal: intuitive understanding of $\hat{L}_\gamma$ oper.

Lindblad eqn. is invariant under:

i) $\hat{L}_\gamma \to \hat{L}_\gamma' = \sum_\alpha U_{\gamma\alpha} \hat{L}_\alpha$ where $U_{\gamma\alpha}$ are components of unitary matrix

$\Gamma = N^2$

ii) $\hat{L}_\gamma \to \hat{L}_\gamma' = L_\gamma + c_\gamma$
$\uparrow$ complex #

$\# \hat{L}_\gamma = N^2-1$

$$\hat{\mathcal{H}}_s \to \hat{\mathcal{H}}_s' = \hat{\mathcal{H}}_s - \frac{i\hbar}{2}\sum_\gamma \left(c_\gamma^* \hat{L}_\gamma - c_\gamma \hat{L}_\gamma^\dagger\right)$$

$U = (N^2-1)\times(N^2-1)$ matrix

use this freedom to choose a set of $\hat{L}_\gamma$ w/ following properties:

$\mathrm{tr}(\hat{L}_\gamma) = 0$ for all γ

[by choosing c_α]

$\mathrm{tr}(\hat{L}_\gamma \hat{L}_\alpha^\dagger) = a_\gamma \delta_{\gamma\alpha}$ [by choosing U]

$a_\gamma \geq 0$

$\uparrow$ Hilbert-Schmidt inner product $\mathrm{tr}(\hat{A}\hat{B}^\dagger)$

write: $\hat{L}_\gamma = \sqrt{a_\gamma}\,\hat{\Lambda}_\gamma \Rightarrow \mathrm{tr}(\hat{\Lambda}_\gamma \hat{\Lambda}_\alpha^\dagger) = \delta_{\gamma\alpha}$

$\mathrm{tr}(\hat{\Lambda}_\gamma) = 0$

unique set of $\hat{\Lambda}_\gamma$ that satisfy these:

basis of Hilbert space

$\{|i\rangle\}$

N-dim.

N^2-1 operators

two types of operators

i) $\hat{\Lambda}_\gamma = |i\rangle\langle j|$ $\quad i \neq j$

"jump" $j \to i$ $\quad$ $N(N-1)$ diff. jump operators

ii) $\hat{\Lambda}_\gamma = \dfrac{\hat{I} - N|i\rangle\langle i|}{\sqrt{N}}$ $\quad i = 2, 3, \ldots, N$

"dephasing" operators $\quad$ $N-1$ diff. operators

concrete example: spin $\frac{1}{2}$ particle interacting w/ env.

$N = 2$ $\quad$ $|\uparrow\rangle = |1\rangle$

$|\downarrow\rangle = |2\rangle$

$N^2 - 1 = 3$ $\quad$ Lindblad operators $\hat{\Lambda}_\gamma$

$\hat{\Lambda}_1 = |1\rangle\langle 2| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ in $\{|1\rangle, |2\rangle\}$ basis

$\hat{\Lambda}_2 = |2\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\hat{\Lambda}_3 = \frac{1}{\sqrt{2}}\left(\hat{I} - 2|2\rangle\langle 2|\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

coefficients: $\quad a_1 = W_{12}$ $\quad a_2 = W_{21}$ $\quad a_3 = \Gamma_2$

$\hat{L}_1 = \sqrt{W_{12}}\,\hat{\Lambda}_1$ $\quad \hat{L}_2 = \sqrt{W_{21}}\,\hat{\Lambda}_2$ $\quad \hat{L}_3 = \sqrt{\Gamma_2}\,\hat{\Lambda}_3$

assume $\hat{H}_S$ is diag. in $\{|1\rangle, |2\rangle\}$

$$\hat{\mathcal{H}}_S = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

a) take $\langle i |$ Lindblad equ. $| i \rangle$:

$$\langle i | \frac{\partial \hat{\rho}}{\partial t} | i \rangle = \langle i | \text{RHS} | i \rangle \qquad i = 1, 2$$

algebra $\rightsquigarrow$

$$\frac{\partial \rho_{ii}}{\partial t} = \sum_{j=1}^{2} \Big[\underbrace{W_{ij} \rho_{jj}}_{\text{gain of } \rho_{ii} \text{ prob.}} - \underbrace{W_{ji} \rho_{ii}}_{\text{loss of } \rho_{ii} \text{ prob.}} \Big]$$

ρ_{ii} = diag. elem of dens. oper.

cont. time version of classical master equ.

$$\left.\begin{aligned} \sum_i \rho_{ii} &= 1 \\ \rho_{ii} &\geq 0 \end{aligned}\right\} \text{behave like classical prob.}$$

$W_{ij} \sim$ "trans. rate of j into i"

$$1 \underset{W_{12}}{\overset{W_{21}}{\rightleftarrows}} 2$$

$$\frac{\partial}{\partial t} \rho_{11} = -W_{21} \rho_{11} + W_{12} \rho_{22}$$

$$\frac{\partial}{\partial t} \rho_{22} = +W_{21} \rho_{11} - W_{12} \rho_{22}$$

solution: $i = 1, 2$

$$\rho_{ii}(t) = \rho_{ii}^S + \left(\rho_{ii}(0) - \rho_{ii}^S \right) e^{-t/T_1}$$

$$\rho_{11}^S = \frac{W_{12}}{W_{12} + W_{21}} \qquad \rho_{22}^S = \frac{W_{21}}{W_{12} + W_{21}}$$

$$T_1 = \frac{1}{W_{12} + W_{21}}$$

relaxation time to stationary state

$\rho_{ii}^S = \rho_{ii}(t \to \infty)$ = stationary state.