Focus: stationary probability $\vec{p}^s = \begin{pmatrix} p_1^s \\ p_2^s \\ \vdots \end{pmatrix}$

$$\vec{p}^s = W \vec{p}^s$$

$$\vec{p}^s = W^2 \vec{p}^s = \dots\dots = W^n \vec{p}^s$$

$\vec{p}^s$ e-vec w/ e-val 1 of W

Question #1: does W always have <u>at least</u> one e-vec w/ e-val 1?

quick useful lemma: matrix M

right e-vec: $M\vec{v} = \lambda \vec{v}$

$\Leftrightarrow$ λ is a sol'n of $\det(M - \lambda I) = 0$

left e-vec: $\vec{u}^T M = \sigma \vec{u}^T$

take transpose: $M^T \vec{u} = \sigma \vec{u}$

$\Rightarrow$ σ is a sol'n of

$$\det(M^T - \sigma I) = 0$$

det A^T = det A

$$\det((M - \sigma I)^T) = 0$$

$$\det(M - \sigma I) = 0$$

$\Rightarrow$ left e-vals σ are <u>same</u> as right e-vals

$\Rightarrow$ if we know a left e-val σ there must exist a right e-vec $\vec{v}$ such that $M\vec{v} = \sigma \vec{v}$

$M = W \Rightarrow$ claim there always exists a left e-vec $\vec{u}^T = (1\ 1\ 1\ \cdots\ 1)$ w/ e-val 1

$$(1\ 1\ 1)\begin{pmatrix} 0.1 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.1 \\ 0.6 & 0.3 & 0.8 \end{pmatrix} = (1\ 1\ 1) \Rightarrow$$ works b/c of columns summing to 1

$\downarrow$

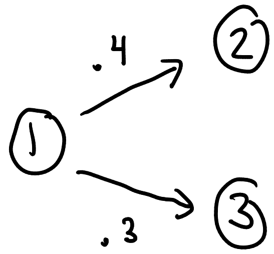
columns sum to 1

$\Rightarrow$ there always exist a right e-vec $\vec{p}^s$ w/ e-val 1 (not necessarily unique)

$$\begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

example:

\# 1 :



$$W = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \ \text{start} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .3 & 0 & 0 \\ .4 & 1 & 0 \\ .3 & 0 & 1 \end{pmatrix} \\ \text{end} & \end{array}$$

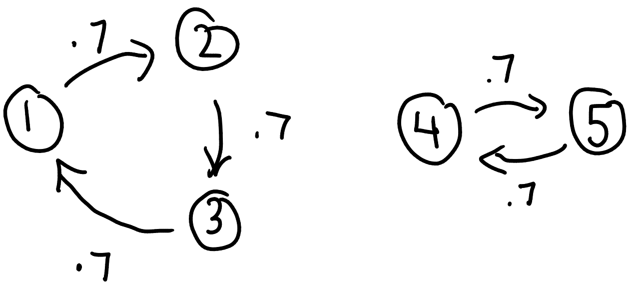
$$W\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{p}_A^{\,s} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{p}_B^{\,s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$W\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{p}_C^{\,s} = \alpha\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1-\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$0 \le \alpha \le 1$ also works

(∞ # of possible stationary states)

2



$$W = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \left(\begin{array}{ccc|cc} .3 & 0 & .7 & 0 & 0 \\ .7 & .3 & 0 & 0 & 0 \\ 0 & .7 & .3 & & \\ \hline 0 & 0 & 0 & .3 & .7 \\ 0 & 0 & 0 & .7 & .3 \end{array}\right) \end{array}$$

disconnected groups $\Rightarrow$ block-diag. W

$$\vec{p}_A^{\,s} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \qquad \vec{p}_B^{\,s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\vec{p}_C^{\,s} = \alpha\, \vec{p}_A^{\,s} + (1-\alpha)\, \vec{p}_B^{\,s} \qquad 0 \le \alpha \le 1$$

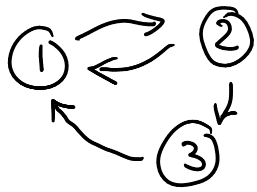
also works

Restrict focus to systems w/ unique $\vec{p}^{\,s}$:

- always consider connected graphs (avoid example #2)
- demand our graph is strongly connected:

  starting from any state you can reach any other state following arrows (avoid example #1)

example :



strongly connected $\equiv$ ergodic graph

as $t \to \infty$ you will visit all states

- final condition (prove this later for classical + quantum systems):

  microscopic reversibility (MR)

  if $W_{ij} \neq 0 \Rightarrow W_{ji} \neq 0$ (either no arrows b/t i + j or double arrows)

  $W_{ij} \neq 0$: $j \to i$ arrow exists

  $W_{ji} \neq 0$: $i \to j$ arrow also exists

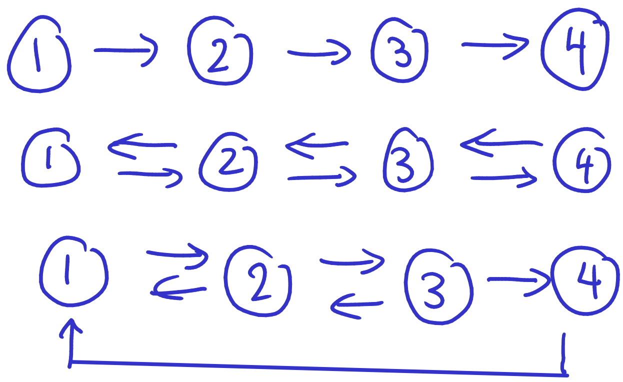
categories:

- all graphs
  - connected
    - ergodic
      - $\boxed{\text{MR}}$
      - non-MR
    - non-ergodic
  - disconnected

connected, non-ergodic:

" ergodic, MR:

" ergodic, non-MR:



Sketch out next steps:

will prove MR graphs have unique $\vec{p}^{\,s}$

to do this we will introduce a useful tool

mean hitting time = avg # of time steps δt to get to state j given starting point at i

$= h_{ji}$ ($< \infty$ for ergodic graphs)