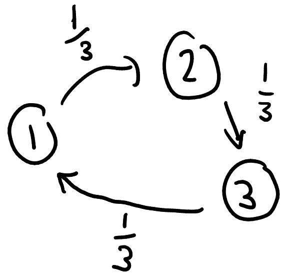
mean hitting time $\equiv h_{ji}$ = avg # of time steps δt to get to state j given start at i

end (j), start (i)

example:



trips b/t 1 and 2

trips b/t 1 and 1

length 2

trajectory: 1 1 2 2 3 3 3 1 2 3 1 1 1 2

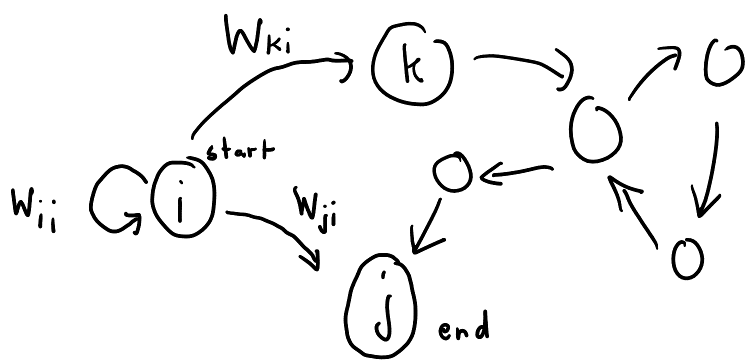
1 1

7 6

h_{21} = avg. of all trip lengths from 1 to 2

h_{11} = " " " " 1 to 1

N^2 quantities h_{ji} $\Rightarrow$ N^2 equations?



$$h_{ji} = W_{ji}\cdot \underset{\substack{\uparrow \\ \text{trip length}}}{1} + \sum_{k\neq j} W_{ki}\,(1+\underbrace{h_{jk}}_{\text{mean trip length}})$$

$$\Rightarrow \quad h_{ji} = \underbrace{W_{ji} + \sum_{k\neq j} W_{ki}}_{=\sum_k W_{ki} = 1} + \sum_{k\neq j} W_{ki}\, h_{jk}$$

$$\boxed{h_{ji} = 1 + \sum_{k\neq j} h_{jk} W_{ki}}$$

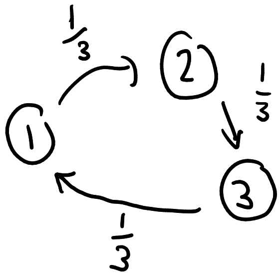
Eq. (*)

$i = 1, \dots, N$

$j = 1, \dots, N$

N^2 equations for N^2 unknowns h_{ji}

example :



$$W = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} \end{array}$$

$i=1,\ j=1:$ $\quad h_{11} = 1 + \sum_{k\neq 1} h_{1k} W_{k1} = 1 + h_{12} W_{21}$

$i=2,\ j=1:$ $\quad h_{12} = 1 + h_{12} W_{22} + h_{13} W_{32}$

$i=3,\ j=1:$ $\quad h_{13} = 1 + h_{13} W_{33}$

$\Rightarrow h_{13} = 3 \qquad h_{12} = 6 \qquad h_{11} = 3$

note: for an ergodic net (path via arrows b/t any two states) $h_{ji} < \infty$ for any i, j

Proof of unique of $\vec{p}^{\,s}$ in an ergodic net:

1) given a W, choose any $\vec{p}^{\,s}$ (at least one exists)

2) multiply both sides of Eq. * by p_i^s:

$$p_i^s h_{ji} = p_i^s \left(1 + \sum_{k\neq j} h_{jk} W_{ki}\right)$$

3) sum both sides over i:

$$\sum_i p_i^s h_{ji} = 1 + \sum_i p_i^s \sum_{k\neq j} h_{jk} W_{ki}$$

$$p_j^s h_{jj} + \sum_{i\neq j} p_i^s h_{ji} = 1 + \sum_{k\neq j} h_{jk} \underbrace{\sum_i W_{ki} p_i^s}_{p_k^s \text{ b/c } W\vec{p}^{\,s} = \vec{p}^{\,s}}$$

$$p_j^s h_{jj} + \sum_{i \neq j} p_i^s h_{ji} = 1 + \sum_{k \neq j} p_k^s h_{jk}$$

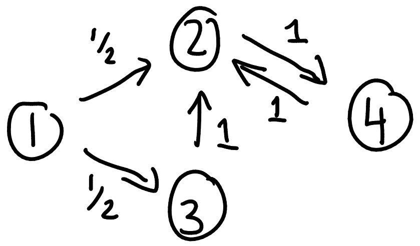
same w/ diff. dummy vars

$$\Rightarrow \quad p_j^s h_{jj} = 1 \Rightarrow \boxed{p_j^s = \frac{1}{h_{jj}}}$$

> 0 b/c $h_{jj} < \infty$ in an ergodic net

$\Rightarrow$ unique solution for $\vec{p}^{\,s}$

Example: Google Page Rank algorithm

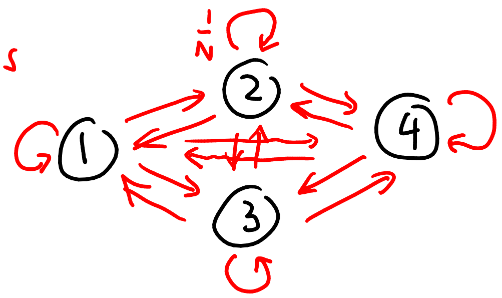


$\longrightarrow$ = hyperlink

(i) = webpage

W^H = trans. matrix created by assigning equal. prob. to all outgoing hyperlinks from a webpage

N = # pages



W^c = trans. matrix of red arrows

$W^c_{ij} = \frac{1}{N}$ for all i, j

$$W = \alpha W^H + \beta W^c$$

$\alpha + \beta = 1$

$\alpha = .85$

$\beta = .15$

guarantee that W network is ergodic $\Rightarrow$ calculate $p_j^s = \frac{1}{h_{jj}}$

& use it to rank webpages

How to efficiently calculate $\vec{p}^{\,s}$?

$\Rightarrow$ not using N^2 equ's for h_{ji}

$\Rightarrow$ better approach: starting w/ arbitrary $\vec{p}^{\,0}$ + use:

$$W \cdots\cdots W W \vec{p}^{\,0} \approx \vec{p}^{\,s}$$

Final puzzle piece: can we prove $W^n \vec{p}^{\,0} \to \vec{p}^{\,s}$ as $n \to \infty$ for any initial $\vec{p}_0$?