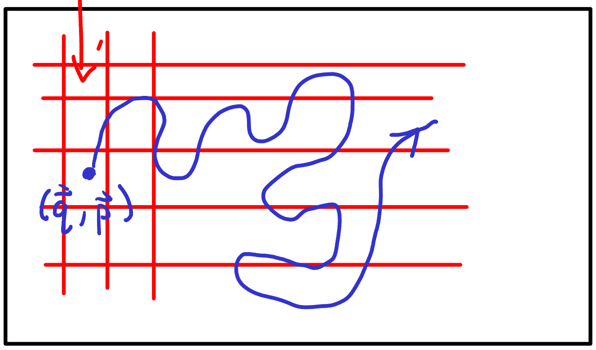


stat. mech focuses on systems w/ certain properties:

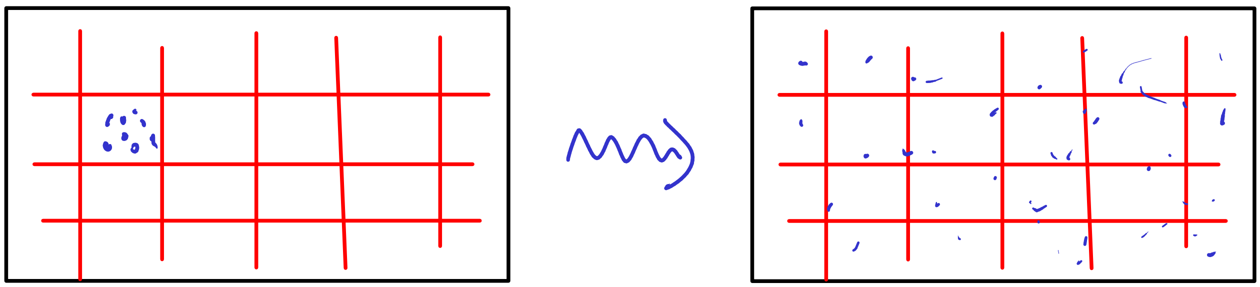
- ergodicity: for any traj. as $t \to \infty$ every microstate μ is visited (no matter how small a is)

  note: in systems w/ nontrivial conserved quantities besides E this may be violated

  in 2d harm. oscill: $E = E_x + E_y$

  $$\left.\begin{aligned} E_x &= \tfrac{1}{2} p_x^2 + \tfrac{1}{2} x^2 \\ E_y &= \tfrac{1}{2} p_y^2 + \tfrac{1}{2} y^2 \end{aligned}\right\} \text{separately conserved}$$

- mixing: if we initiate a group of traj. at $t = 0$ in same microstate μ
  $\Rightarrow$ traj. spread out evenly to all microstates as $t \to \infty$



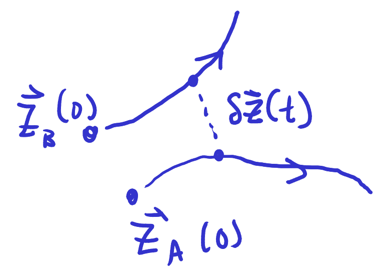
$t=0$ $\qquad\qquad$ $t\to\infty$

ergodicity $\subset$ mixing

How is this possible?

Two traj. w/ diff. initial conditions:

$$\vec{z} = (\vec{q}, \vec{p})$$



$$\vec{z}^A(t) \qquad\qquad \vec{z}^B(t)$$

$$\delta\vec{z}(t) = \vec{z}^A(t) - \vec{z}^B(t)$$

Chaos: $\quad |\delta\vec{z}(t)| \sim e^{\lambda t}|\delta\vec{z}(0)|$ $\quad$ for initial times t small $|\delta\vec{z}(0)|$

w/ $\lambda > 0$

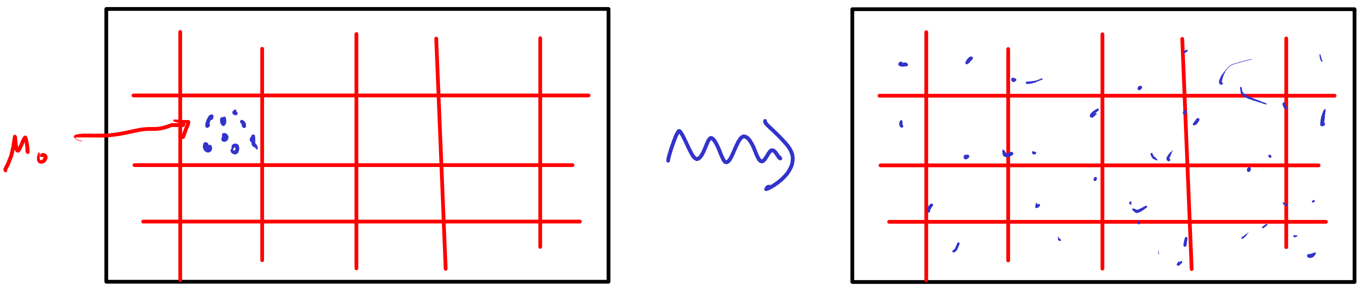
Lyapunov exponent

plausibly this chaotic behavior (at short times) could lead to mixing (at long times) $\Rightarrow$ but hard to prove rigorously!

if mixing is true:

$P_\mu(t)$ = prob. to be in microstate μ at time t

$\Theta(E)$ = # boxes



$t = 0$ $\qquad$ $t \to \infty$

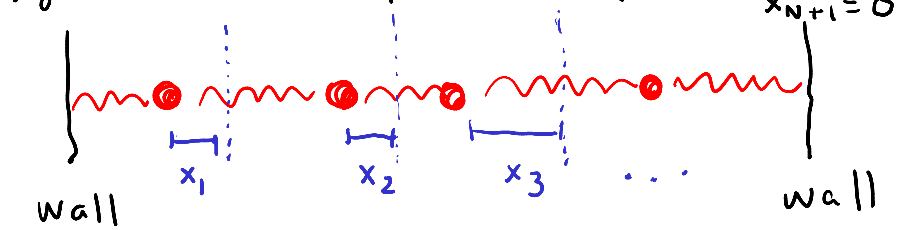
$$P_\mu(0) = \begin{cases} 1 & \text{if } \mu = \mu_0 \\ 0 & \text{if } \mu \neq \mu_0 \end{cases} \rightsquigarrow P_\mu(t) = \frac{1}{\Theta(E)}$$

all microstates equally likely: microcanonical ensemble

First test: 1953-55 Fermi, Pasta, Ulam, Tsingou (FPUT): first physics computer experiment

$x_0 = 0$ $\qquad$ $x_{N+1} = 0$

connected springs:



N masses: $m = 1$

w/ spring constants $k = 1$

Hookean spring forces

eq. of motion: $$\ddot{x}_n = \underbrace{\left[(x_{n+1} - x_n) - (x_n - x_{n-1})\right]}_{\text{Hookean spring forces}} \cdot \left(1 + \alpha (x_{n+1} - x_{n-1})\right)$$

$\alpha \neq 0$: nonlinear perturb. to Hooke's law

$\underline{\alpha = 0}$ $\mathcal{H}(\vec{x}, \vec{p}) = \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + \frac{(x_n - x_{n-1})^2}{2} \right) + \frac{x_N^2}{2}$

$\vec{x} = (x_1, \ldots, x_n)$ $\quad p_n = \dot{x}_n$

$\vec{p} = (p_1, \ldots, p_n)$

reminder: any funcs $f(\vec{x}, \vec{p})$ $g(\vec{x}, \vec{p})$

Poisson brackets: $\{f, g\}_{\vec{x}, \vec{p}} \equiv \sum_{n=1}^{N} \left(\frac{\partial f}{\partial x_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial x_n} \right)$

↑ left out of notation

$\Rightarrow$ can show: $\frac{d}{dt} f = \{f, H\}$

$f = x_i \Rightarrow \dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ $\quad f = p_i \Rightarrow \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}$

$\{x_i, x_j\} = 0 \quad \{p_i, p_j\} = 0 \quad \{x_i, p_j\} = \delta_{ij}$

---

canonical transf. (CT) of coord:

define $\vec{Q}(\vec{x}, \vec{p})$ $\vec{\mathcal{P}}(\vec{x}, \vec{p})$

that preserve Poisson brackets:

$\{f, g\}_{\vec{x}, \vec{p}} = \{f, g\}_{\vec{Q}, \vec{\mathcal{P}}}$

$\Rightarrow$ leads: $\dot{Q}_n = \{Q_n, \mathcal{H}\}_{\vec{Q}, \vec{\mathcal{P}}} = \frac{\partial \mathcal{H}}{\partial \mathcal{P}_n}$

$\dot{\mathcal{P}}_n = \{\mathcal{P}_n, \mathcal{H}\}_{\vec{Q}, \vec{\mathcal{P}}} = -\frac{\partial \mathcal{H}}{\partial Q_n}$