FPUT system: connected springs ($\alpha = 0$) Hookean

canonical transf: define new "coords" + "mom"

"phase" $\phi_i \updownarrow x_i$ $\quad$ $I_i \updownarrow p_i$ "amplitude"

$$x_n = \sum_{k=1}^{N} \sqrt{\frac{2 I_k}{(N+1)\omega_k}} \sin\left(\frac{nk\pi}{N+1}\right) \sin(\phi_k)$$

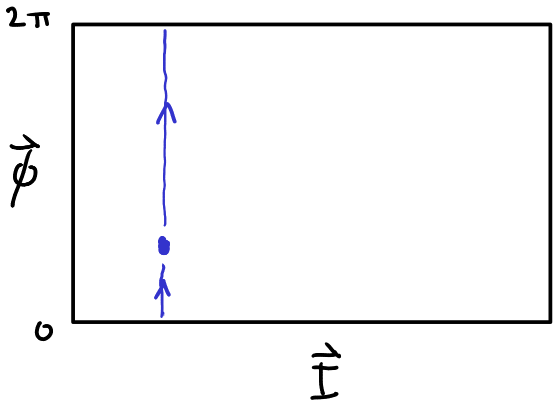
$$p_n = \sum_{k=1}^{N} \sqrt{\frac{2 I_k \omega_k}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right) \cos(\phi_k)$$

$$\omega_k = 2 \sin\left(\frac{k\pi}{2(N+1)}\right) \qquad k = 1, \ldots, N$$

check: $\{x_i, p_j\}_{\vec{\phi}, \vec{I}} = \delta_{ij}$, etc.

after transf: $$\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \underbrace{\overset{\text{constants}}{\omega_k} I_k}_{\text{energy of } k\text{th "normal mode"}}$$

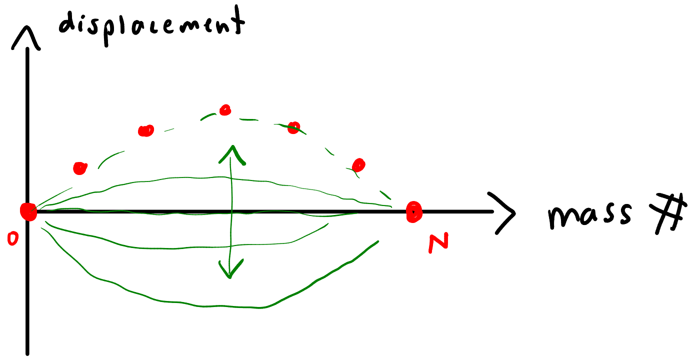
phase space



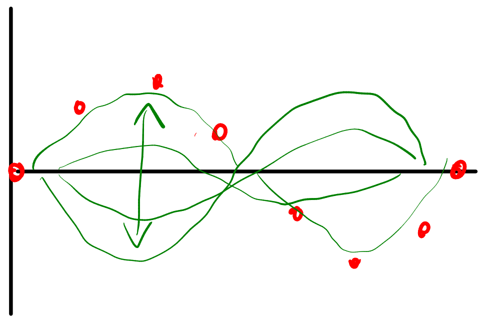
$$\dot{\phi}_i = \frac{\partial \mathcal{H}}{\partial I_i} = \omega_i \Rightarrow \phi_i(t) = \omega_i t + \phi_i(0)$$

$$\dot{I}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i} = 0 \Rightarrow \text{all } I_i \text{ are constants of motion}$$

$k=1$



$k=2$



What happens when $\alpha \neq 0$?

(non-Hookean springs)

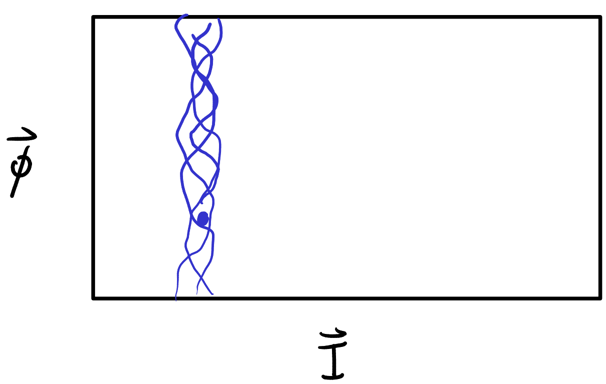
$$\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \omega_k I_k + \alpha U(\vec{\phi}, \vec{I}) + \dots$$

for small α

$$\Rightarrow \quad \dot{I}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i} \neq 0$$

I_i are no longer constants of motion!

$\alpha \neq 0$ case



$\alpha \neq 0$: FPUT observation: quasiperiodic behavior where you return close to (not exactly) your initial conditions

$\Rightarrow$ never achieve microcanonical ensemble (mixing) expected by Fermi

Consider a more general class of systems:

$\mathcal{H}(\vec{x}, \vec{p})$ $\quad \vec{x} = (x_1, \dots, x_n)$ $\quad$ 2n-dim. phase space

$\vec{p} = (p_1, \dots, p_n)$

This system is integrable when:

i) there are n linearly indep. consts of motion: $F_k(\vec{x}, \vec{p})$ $k = 1, \dots, n$

$$\Rightarrow \quad \frac{dF_k}{dt} = \{F_k, \mathcal{H}\} = 0$$

by convention: $F_1 \equiv \mathcal{H}$

ii) $\{F_k, F_\ell\} = 0$ for all k, ℓ

Liouville-Arnold theorem: for these integrable systems there exists a canonical transf. to "action-angle" coords:

$$(\vec{\phi}, \vec{I})$$

"angles" $(\phi_1, \dots, \phi_n)$

"actions" $(I_1, \dots, I_n)$

note: $\phi_i + 2\pi \underbrace{m}_{\text{integer}} \equiv \phi_i$

$$\mathcal{H} = \mathcal{H}(I)$$

$$\dot{I}_k = -\frac{\partial \mathcal{H}}{\partial \phi_k} = 0 \quad \text{all } I_k \text{ are consts. of motion}$$

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial I_k}(\vec{I}) \equiv \omega_k(\vec{I})$$

$$\phi_k(t) = \omega_k(I)\, t + \phi_k(0)$$

examples of integrable systems:

- all 1D problems w/ conserved energy
- n coupled harmonic springs
- central force problems
- two-body grav. problems
- gyroscopes + tops
- free particles confined on surfaces of ellipsoids

not integrable:

- three-body problem (Poincaré)
- chaotic systems
- dissipative systems