RECAP: coord transf $\Rightarrow$ $(\vec{\phi}, \vec{I})$

integrable systems : $\mathcal{H} = \mathcal{H}(I)$

$$\dot{I}_k = -\frac{\partial \mathcal{H}}{\partial \phi_k} = 0$$

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial I_k}(\vec{I}) \equiv \omega_k(\vec{I})$$

integrate eqs of motion: $I_k = \text{constant}$

$$\phi_k = \omega_k(\vec{I})t + \phi_k(0)$$

$$\phi_k + 2\pi = \phi_k$$

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geometrical interpretation:

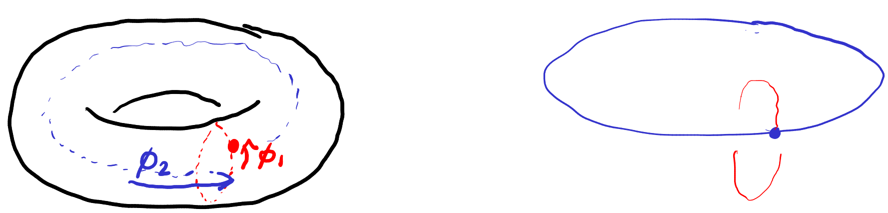
for integrable system, everything "lives" on a torus (donut)

$$\vec{\phi}(t) = (\phi_1, \phi_2, \ldots, \phi_n)$$ coords on T^n

space $T^n = S^1 \times S^1 \times \ldots S^1$

"hypertorus" circle

$n=2$:



convention: I_1 I_2 : two "radii" of torus

$x_1, \dots, x_n \quad p_1, \dots, p_n$

$(\vec{x}, \vec{p}) \Rightarrow (\vec{\phi}, \vec{I})$

all phase space

all of phase space

each choice of $\vec{I}$ corresponds to one torus + set of all torii $\Rightarrow$ all phase space

integrable system: dynamics stay on one torus, one "page" in the "foliation"

"foliation" of phase space

classify tori:

i) resonant tori:

there exists a vector $\vec{v}$ of integers $\vec{v} \neq 0 \quad v_i \in \mathbb{Z}$

such that $\vec{v} \cdot \vec{\omega} = 0$

$$\vec{\omega} = (\omega_1(\vec{I}), \omega_2(\vec{I}), \dots, \omega_n(\vec{I}))$$

special case: $\omega_i = \underset{\text{integer}}{z_i} \cdot \underset{\text{const.}}{c}$

$\Rightarrow$ periodic, closed orbits on torus

ii) non-resonant tori: no such $\vec{v}$ exists

$\Rightarrow$ never return to orig. position on torus

$\Rightarrow$ trajectories fill up torus densely

$\Rightarrow$ you will return arbitrarily close to initial position

$\Rightarrow$ quasiperiodicity (even if sys. is integrable)

What happens if you break integrability via a small perturbation?

$$\mathcal{H}(\vec{\phi}, \vec{I}) = \mathcal{H}_0(\vec{I}) + \alpha \mathcal{H}_1(\vec{\phi}, \vec{I})$$

original Hamilt. integrable

$\hookrightarrow$ small

Kolmogorov, Arnold, Moser (KAM) theorem:
1954-63

non-technical: for small α, donuts persist

$\Rightarrow$ trajectories live on "deformed" donuts