KAM theorem: "persistence of donuts"

setup: integrable system $\mathcal{H}_0(\vec{I})$

frequencies: $\omega_k(\vec{I}) = \dfrac{\partial \mathcal{H}_0}{\partial I_k}$

$$\phi_k(t) = \omega_k(\vec{I})\,t + \phi_k(0)$$

pick initial conditions:

$(\vec{x}_0, \vec{p}_0) \Rightarrow$ choosing values of $\vec{I}$ (consts. of motion)

$\Rightarrow$ know freq. $\omega_k(\vec{I})$

$\Rightarrow$ all traj. confined to corresponding hypertorus "donut" assoc. w/ $\vec{I}$

Question: what happens when you add a perturbation:

$$\mathcal{H}(\vec{\phi}, \vec{I}) = \mathcal{H}_0(\vec{I}) + \underset{\substack{\uparrow \\ \text{small}}}{\alpha}\, \mathcal{H}_1(\vec{\phi}, \vec{I})$$

Does the donut still exist?

KAM theorem says: We can prove "YES" but only under certain conditions

conditions for proof:

1) torus is <u>strongly</u> non-resonant

RECALL: torus is resonant when there exists a vector of integers $\vec{v} \neq 0$ where $\vec{v} \cdot \vec{\omega} = 0$

$$\vec{\omega} = (\omega_1, \dots, \omega_n)$$

non-resonant: no such vector exists

strongly non-resonant: for any $\vec{v} \neq 0$ vector of integers

$$\vec{v} \cdot \vec{\omega} \geq \frac{\epsilon}{|\vec{v}|^\tau}$$ for some real #'s ϵ + $\tau > n-1$

note: for small ϵ, most tori are strongly non-resonant

2) system is non-degenerate:

$$M_{ij} = \frac{\partial^2 H_0}{\partial I_i \partial I_j} = \frac{\partial \omega_i}{\partial I_j}$$ $\det(M) \neq 0$

$n \times n$ matrix

KAM statement: there $\delta > 0$

such that for perturbations with $\alpha \leq \delta \epsilon^2$ all strongly non-resonant tori survive, + are only slightly deformed

$\Rightarrow$ we still get quasiperiodic trajectories

note: FPUT system is technically degenerate $\Rightarrow$ original KAM proof does not apply

1970s: Nishida conjecture: for low energies, KAM-like behavior for FPUT

2005: Bob Rink proved conjecture for FPUT

↳ arXiv: 0506024

"Donut" classification of physical systems:

behavior of trajectories	classical version	quantum version
• confined to donuts	$\mathcal{H} = \mathcal{H}(\vec{I})$	quantization rule $I_k = \hbar(n_k + c_k)$; n_k: integer ≥ 0; c_k: const. Maslov correction
• confined to deformed donuts	$\mathcal{H} = \mathcal{H}_0(\vec{I}) + \alpha U(\vec{\phi}, \vec{I})$, α small	no simple quantization rules exist
• not confined (no donuts!) chaos	more general Hamiltonians (α no longer small)	

class. mech.

domain of statistical mechanics

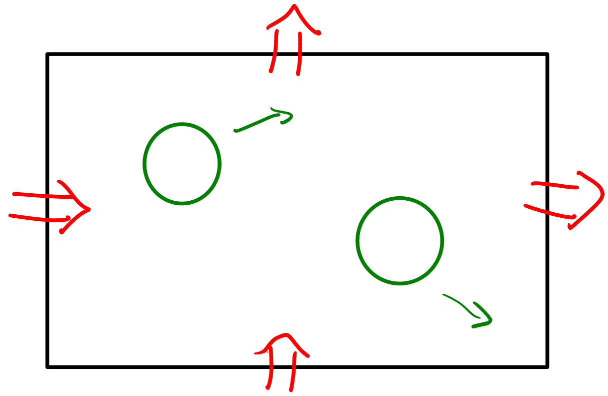
should lead to ergodicity + mixing (traj. visit all of phase space)

1970: first rigorous proof that a system could be ergodic + mixing $\Rightarrow$ Yakov Sinai

Sinai billiard:

hard disks colliding

phase space: all positions & velocities of disks



2D box w/ periodic B.C.

current state-of-art:

$N \geq 2$ d-dim. spheres are almost proven ergodic