Interpretation of entropy as information: Shannon source coding theorem

surprisal: $-k_B \ln p_n(t)$

entropy: avg surprisal $S(t) = -k_B \sum_n p_n(t) \ln p_n(t)$

imagine scenario: Alice (A) is a prof.
Bob (B) is a grad. stud.

experiment: a system ($N=4$ states) (stochastic)

One run:

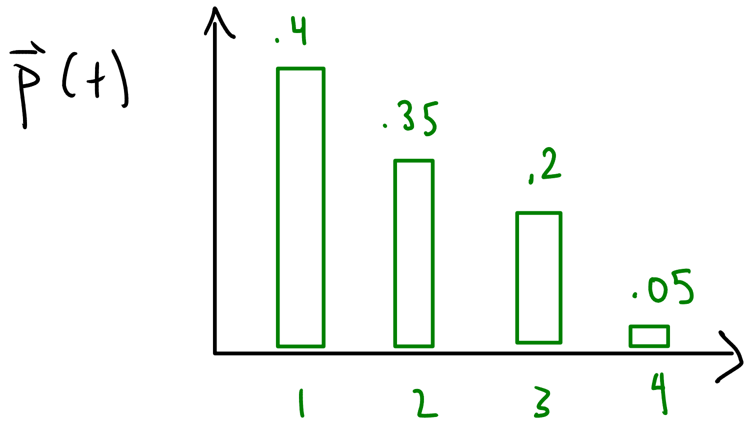
repeat it many times $\begin{cases} \bullet \text{ B prepares it in init. dist. } \vec{p}(0) \\ \bullet \text{ B lets it evolve for time } t \\ \bullet \text{ B takes measurement, records state} \end{cases}$

output: series of measurements:

i.e. 2, 4, 1, 3, 2, ...

A + B know: system trans. matrix W
prob. $\vec{p}(0)$
hence $\vec{p}(t) = W^t \vec{p}(0)$

A doesn't know specific seq. of measurements



case I: Bob is lazy

state	code
1	00
2	01
3	10
4	11

easy to read (every 2 bits is a state)

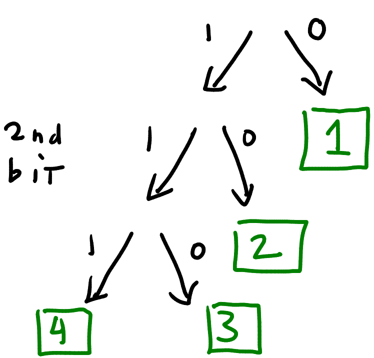
if there were N states in general

mean message length per state

$$= \log_2 N$$

case II: Bob is clever: a variable length code called a "prefix-free" code: no stop markers b/t states

coding / decoding $\Longleftrightarrow$ binary decision tree



one example (not unique)

state	code	# bits ℓ_n
1	0	1
2	10	2
3	110	3
4	111	3

Seq: 2 1 4 3 $\Rightarrow$ 100111110 ....

avg. # bits per state (cost of sending code)

$$B = \sum_n p_n(t) \ell_n = 0.4 \times 1 + .35 \times 2 + 0.2 \times 3 + .05 \times 3$$

$$= 1.85 \text{ bits / state}$$

for the above code

Shannon proved: among all possible codes (all possible binary decision trees) there is a lower bound:

$$B \geq B_{min} = -\sum_n p_n(t) \log_2 p_n(t)$$

"information entropy"

units: bits

compare to thermo. entropy

$$S(t) = -k_B \sum_n p_n(t) \ln p_n(t)$$

$$B_{min} = \frac{S(t)}{k_B \ln 2}$$

converts from $\frac{J}{K}$ to bits

examples: $p_n(t) = \delta_{n,1}$ all states are 1 in the message

$B_{min} = 0$ (no info needs to be sent)

$$P_n(t) = \frac{1}{4}$$ (all states equally likely in message)

$$B_{min} = 2 \text{ bits}$$ (can't do any better than lazy strategy)