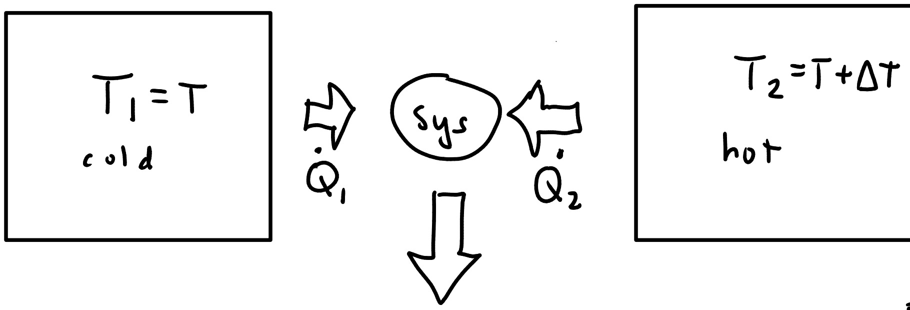
van der Broeck PRL (2005);



$$P = f\dot{x}$$

$$\eta = \frac{P}{\dot{Q}_2} \leq \eta_{carnot} = 1 - \frac{T_1}{T_2} \approx \frac{\Delta T}{T}$$

$$\begin{pmatrix} J_H \\ J_M \end{pmatrix} = \begin{pmatrix} \dot{Q}_2 \\ -\dot{x} \end{pmatrix} = \underbrace{\begin{pmatrix} L_{HH} & L_{HM} \\ L_{MH} & L_{MM} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \frac{\Delta T}{T^2} \\ \frac{f}{T} \end{pmatrix}}_{\begin{pmatrix} Q_H \\ Q_M \end{pmatrix}}$$

$$\vec{J} = L\vec{\phi}$$

$\sigma = \vec{\phi}^T L \vec{\phi} \geq 0$ true for any (small) $\vec{\phi}$

What do we know about L?

1) symmetric : $L_{MH} = L_{HM}$

$\Rightarrow$ eigenvalues are real: λ_1, λ_2

$\Rightarrow$ e-vecs are orthonormal: $\vec{u}^{(i)} \cdot \vec{u}^{(j)} = \delta_{ij}$

2) plug in $\vec{\phi} = \underset{\uparrow \text{ small}}{\varepsilon} \vec{u}^{(i)}$

$$\sigma = \varepsilon^2 \vec{u}^{(i)T} L \vec{u}^{(i)} = \varepsilon^2 \lambda_i \vec{u}^{(i)T} \vec{u}^{(i)}$$

$$= \varepsilon^2 \lambda_i \geq 0$$

$$\Rightarrow \lambda_i \geq 0$$

$\Rightarrow$ all e-vals ≥ 0

3) $\det L = \lambda_1 \lambda_2 \geq 0$

4) plug in $\vec{\phi} = \varepsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad \sigma = \vec{\phi}^T L \vec{\phi} = \varepsilon^2 L_{ii} \geq 0 \Rightarrow L_{ii} \geq 0 \text{ for all } i$$

$\Rightarrow$ diag. elements of L are pos.

summary: $L_{HH}, L_{MM} \geq 0 \qquad L_{HM} = L_{MH}$

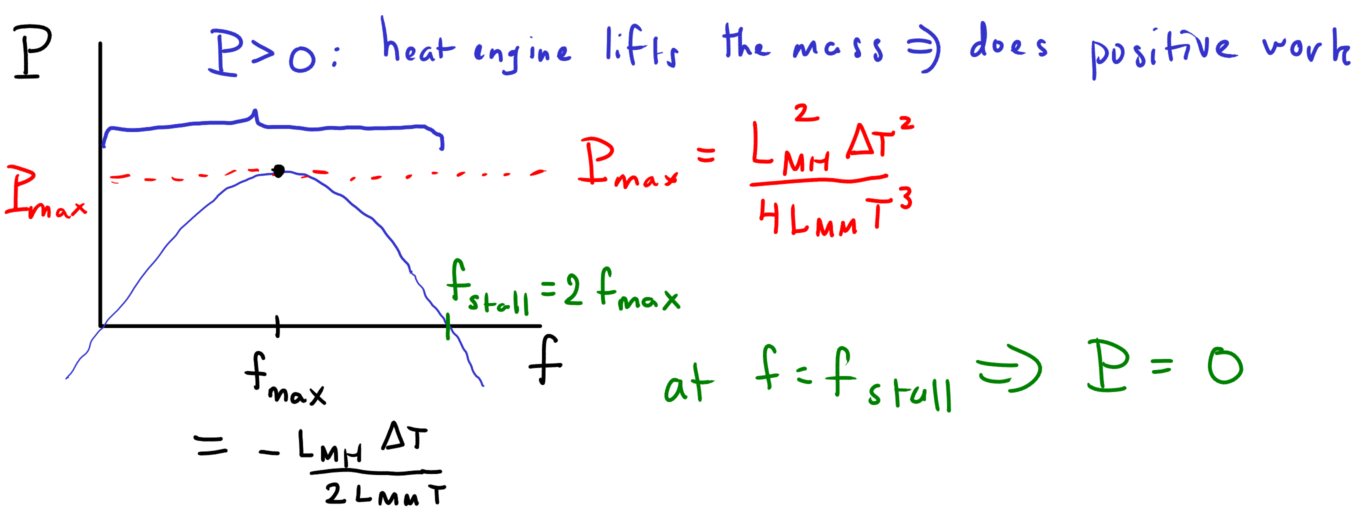
$$\det(L) = L_{HH} L_{MM} - L_{HM} L_{MH} \geq 0$$

$$P = f\dot{x} = -f\left(L_{MH}\phi_H + L_{MM}\phi_M\right)$$

$$= -L_{MM}\frac{f^2}{T} - L_{MH}\frac{\Delta T f}{T^2}$$

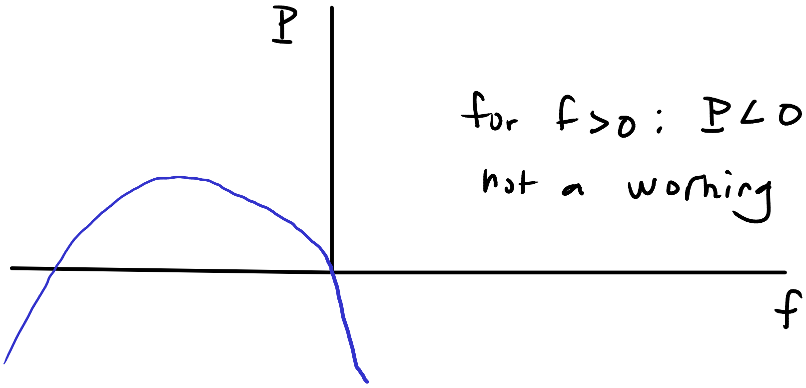
$$= -\frac{L_{MM}}{T}\left(f + \frac{L_{MH}\Delta T}{2 L_{MM} T}\right)^2 + \frac{L_{MH}^2 \Delta T^2}{4 L_{MM} T^3}$$

case 1: $L_{MH} = L_{HM} < 0$



at $f = f_{stall} \Rightarrow P = 0$

Case 2: $L_{MH} = L_{HM} > 0$



for $f > 0$: $P < 0$
not a working engine

$$\vec{J} = L \vec{\phi}$$

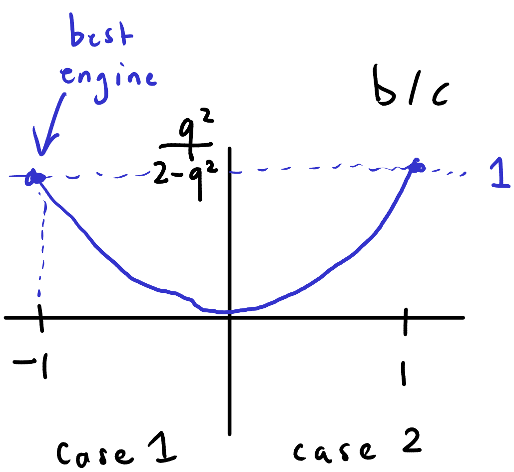
focus on case 1:

$$\dot{Q}_2 = J_H = L_{HH} \phi_H + L_{HM} \phi_M$$

$$= L_{HH} \frac{\Delta T}{T^2} + L_{HM} \frac{f}{T}$$

at $f = f_{max}$: $$= \frac{\Delta T}{T^2} \left(L_{HH} - \frac{L_{MM} L_{MH}}{2 L_{MM}} \right)$$

$$\eta = \frac{P_{max}}{\dot{Q}_2 (f = f_{max})} = \frac{\Delta T}{2T} \frac{q^2}{2 - q^2} \qquad q \equiv \frac{L_{MH}}{\sqrt{L_{MM} L_{HH}}}$$



b/c $\det(L) \geq 0 \Rightarrow q^2 \leq 1$

$$-1 \leq q \leq 1$$

$q < 0$ case 1
$q > 0$ case 2

$$\eta \leq \frac{\Delta T}{2T}$$

at max. power

max. efficiency at max. power

compare $\eta \leq \frac{\Delta T}{T}$ Carnot efficiency

$$\equiv \eta_{carnot}$$

at max. power $\eta \leq \frac{1}{2} \eta_{carnot}$

Quantum stat. mechanics

ensemble: many copies of system prepared at $t=0$

- classical: $p_n(0) =$ frac. of ensemble in classical state n at $t=0$
- quantum: $p_n(0) =$ " " " in quantum state $|\psi_n\rangle$

$\{|\psi_n\rangle\}$ some arbitrary set of states in a Hilbert space: not necessarily orthog. or a complete basis

but we require $\langle\psi_n|\psi_n\rangle = 1$ (norm.)

N_{tot} copies in ensemble:

$$\underbrace{|\psi_1\rangle\ |\psi_1\rangle\ |\psi_1\rangle}_{p_1(0)=\frac{3}{N_{tot}}}\quad \underbrace{|\psi_2\rangle\ |\psi_2\rangle}_{p_2(0)=\frac{2}{N_{tot}}}\ |\psi_3\rangle \cdots \quad \text{etc.}$$

properties: $p_n(0) \geq 0$ $\quad \sum_n p_n(0) = 1$

in classical case: state n had properties $E_n, x_n, \ldots$

$$\langle E \rangle = \sum_n p_n E_n$$

in quantum case, each observable A has an operator $\hat{A}$

avg. of A in state $|\psi_n\rangle$: $\langle \psi_n | \hat{A} | \psi_n \rangle$

avg. over whole ensemble $\quad \langle A \rangle = \sum_n p_n \langle \psi_n | \hat{A} | \psi_n \rangle$

$\hat{A}$ is Hermitian: e-vecs $|a\rangle$

$$\underbrace{\sum_a |a\rangle\langle a| = \hat{I}}_{\text{identity}} \qquad \hat{A}|a\rangle = \underbrace{a}_{\text{e-val}}|a\rangle$$

$$\langle A \rangle = \sum_{n,a} p_n \langle \psi_n | \hat{A} | a \rangle \langle a | \psi_n \rangle \qquad (1)$$

$$= \sum_{n,a} p_n a \langle \psi_n | a \rangle \langle a | \psi_n \rangle$$

$$= \sum_{n,a} a \underbrace{p_n |\langle a | \psi_n \rangle|^2}_{\text{two contrib to prob:}}$$

- p_n: prob to find $|\psi_n\rangle$ in ensemble

- $|\langle a|\psi_n\rangle|^2$: prob. to find result a when measuring $\hat{A}$ in state $|\psi_n\rangle$

rewrite: Eq. (1)

$$\langle A\rangle = \sum_{n,a} p_n \langle a|\psi_n\rangle\langle\psi_n|\hat{A}|a\rangle$$

$$= \sum_a \langle a| \underbrace{\left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right]}_{\equiv\ \hat{\rho}} \hat{A}|a\rangle$$

$\equiv \hat{\rho}$ density operator for our ensemble

$$\boxed{\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n|}$$

density operator: central quantity in quantum stat. mech.

$$\langle A\rangle = \sum_a \langle a|\hat{\rho}\hat{A}|a\rangle = \mathrm{tr}(\hat{\rho}\hat{A})$$

basis independent