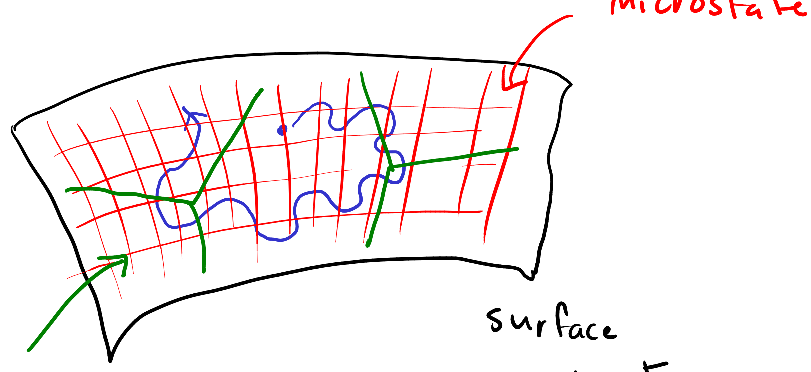
How to define quantum entropy?

recap classical:



$$P_n(t) \Rightarrow S(t) = -k_B \sum_n p_n(t) \ln p_n(t)$$

quantum: goal: assign an entropy to a density operator $\hat{\rho}$

problem: each $\hat{\rho}$ corresponds to potentially an ∞ # of different ensembles (decompositions) w/ diff. prob's

von Neumann solution: choose a special decomposition $\Rightarrow$ orthonormal decomp. (OD)

$\hat{\rho}$ Hermitian $\Rightarrow$ complete basis of e-states $|\phi_n\rangle$

$$\hat{\rho}\,|\phi_n\rangle = \overset{p_n}{\overset{\shortparallel}{\lambda_n}}\,|\phi_n\rangle$$

λ_n ← e-vals (interpret as prob's)

$\Downarrow$

$$\boxed{\hat{\rho} = \sum_n p_n |\phi_n\rangle\langle\phi_n|}$$ OD

note: $\langle \phi_n | \phi_m \rangle = \delta_{nm}$ $\quad tr(\hat{\rho}) = 1 \Rightarrow \sum_n \lambda_n = 1$

$\lambda_n \geq 0$

states $|\phi_n\rangle$ are distinguishable:

a measurement that tells us we are in $|\phi_n\rangle \Rightarrow$ all subs. measurements cannot give $|\phi_m\rangle$ $m \neq n$

$\Rightarrow$ the states "look" classical

k_B usually left out

define von Neumann entropy $S(\hat{\rho}) = -\sum_n p_n \ln p_n$

p_n: e-vals of $\hat{\rho}$

to calculate: i) find $\hat{\rho}$

ii) find e-vals of $\hat{\rho} \Rightarrow \lambda_n = p_n$

iii) calculate $S(\hat{\rho})$

observable: $\hat{A} \Rightarrow$ e-vecs $|a_i\rangle$

$$|\phi_n\rangle = \sum_i c_i |a_i\rangle$$

$$\langle \phi_n | \phi_m \rangle = \delta_{nm}$$

$m \neq n$

$$|\phi_m\rangle = \sum_i c_i' |a_i\rangle$$

state $|\phi_n\rangle \Rightarrow$ measure $A \Rightarrow$ collapse onto one of the $|a_i\rangle$

in $|\phi_n\rangle$ subspace

$\Rightarrow$ remains orthog. to $|\phi_m\rangle$ subspace

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$\hat{\rho} = |\psi\rangle\langle\psi|$ 100% in $|\psi\rangle$

in $|0\rangle, |1\rangle$ basis: $\hat{\rho} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

diagonalize: e-vals: 1, 0

$$S(\hat{\rho}) = -1 \ln 1 - 0 \ln 0$$
$$= 0$$

Time evolution of $\hat{\rho}(t)$:

simplest case: isolated system (no interactions w/ outside) described by a Hamiltonian $\hat{H}(t)$

closed quantum system

wavefunction $|\psi(t)\rangle$:

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

unitary operator: propagator

$$U^{\dagger}(t)\,\hat{U}(t) = \hat{I}$$
$$= \hat{U}(t)\,\hat{U}^{\dagger}(t)$$

special case: $\hat{H}(t) = \hat{H} \Rightarrow \hat{U}(t) = e^{-i\hat{H}t/\hbar}$

ensemble: focus on 0D ensemble

$t = 0$		increasing time	$t > 0$	
frac	state		frac.	state
p_1	$\|\phi_1(0)\rangle$		p_1	$\|\phi_1(t)\rangle = \hat{U}(t)\,\|\phi_1(0)\rangle$
p_2	$\|\phi_2(0)\rangle$		p_2	$\|\phi_2(t)\rangle = \hat{U}(t)\,\|\phi_2(0)\rangle$
$\vdots$	$\vdots$		$\vdots$	

$$\hat{\rho}(t) = \sum_n p_n\,|\phi_n(t)\rangle\langle\phi_n(t)|$$
$$= \sum_n p_n\,\hat{U}(t)\,|\phi_n(0)\rangle\langle\phi_n(0)|\,\hat{U}^{\dagger}(t)$$
$$= \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{\dagger}(t)$$

$$\boxed{\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{\dagger}(t)}$$

dynamics of $\hat{\rho}$ for any closed qu. system note: at $t=0$ we started in OD

claim: at all $t>0$ we will remain in OD

proof: $\langle \phi_n(t) | \phi_m(t) \rangle = \langle \phi_n(0) | \hat{U}^\dagger(t) \hat{U}(t) | \phi_m(0) \rangle$

$$= \langle \phi_n(0) | \phi_m(0) \rangle$$

$$= \delta_{nm}$$

$\Rightarrow$ $\hat{\rho}(t)$ starts diag at $t=0$

remains diag. at $t>0$

$p_1, p_2, \ldots$ remain e-vals at all $t \geq 0$

$$\boxed{S(\hat{\rho}(t)) = -\sum_n p_n \ln p_n = S(\hat{\rho}(0))}$$

Von Neumann entropy is constant in time for closed qu. system

contrast: closed classical system (hockey stadium billiard)

Gibbs entropy $S(t) \xrightarrow[\text{inc. w/ time}]{} S^{eq} = k_B \ln N$

$$\underset{\text{von Neumann}}{S(\hat{\rho})} \underset{\text{class. limit}}{\neq} \underset{(\text{"therm. entropy"})}{\text{Gibbs entropy}}$$