von Neum. entropy: $\hat{\rho} \Rightarrow$ find e-vals p_n

$$S(\hat{\rho}) = -\sum_n p_n \ln p_n$$

closed qu. system: $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$

$$\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{+}(t)$$

$$\Rightarrow \quad S(\hat{\rho}(t)) = S(\hat{\rho}(0))$$

---

next steps: add interactions w/ outside env.

$\Rightarrow$ $\hat{\rho}(t)$ generally evolves according Choi-Krauss repr. theorem

$\Rightarrow$ quantum master equ. for $\hat{\rho}(t)$

simplest interactions: measurements

initial ensemble: <u>frac</u> p_1 p_2 . . . . .

<u>state</u> $|\psi_1\rangle$ $|\psi_2\rangle$ . . . . . .

measure $\hat{A}$

w/ e-states

$\hat{A}\,|a\rangle = a\,|a\rangle$

collapse to one of $|a\rangle$ w/ prob $|\langle a|\psi_1\rangle|^2$

" " " $|\langle a|\psi_2\rangle|^2$

final ensemble	frac.	p_a	p_b	....
	state	$\|a\rangle$	$\|b\rangle$	.....

$$p_a = \sum_n p_n |\langle a|\psi_n\rangle|^2$$

p_n ← frac. of $|\psi_n\rangle$ in orig. ensemble

new density op: $\hat{\rho}' = \sum_a p_a |a\rangle\langle a|$

$$= \sum_{a,n} p_n \langle a|\psi_n\rangle\langle\psi_n|a\rangle |a\rangle\langle a|$$

$$= \sum_a |a\rangle\langle a| \underbrace{\left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right]}_{\hat{\rho}} |a\rangle\langle a|$$

proj. operator

$\hat{P}_a = |a\rangle\langle a|$

$\hat{P}_a^{\dagger} = |a\rangle\langle a|$

$$\boxed{\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^{\dagger}}$$

dynamics after measurement

compare to: $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^{\dagger}$ dynamics for closed sys.

example: initially 100% in state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\hat{\rho} = |\psi_1\rangle\langle\psi_1|$$

$S(\hat{\rho}) \Rightarrow$ write $\hat{\rho}$ in a basis

diag + find e-vals

basis: $\left\{ |\psi_1\rangle ,\ |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \right\}$

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{e-vals} \quad \begin{aligned} p_1 &= 1 \\ p_2 &= 0 \end{aligned}$$

$$S(\hat{\rho}) = -\sum_n p_n \ln p_n = 0 \qquad \text{pure state}$$

measurement:

$$|\psi_1\rangle \begin{cases} \xrightarrow[\text{collapse}]{50\%} |0\rangle \\ \xrightarrow{50\%} |1\rangle \end{cases}$$

$$\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= |0\rangle \underbrace{\langle 0|\hat{\rho}|0\rangle}_{1/2} \langle 0| + |1\rangle \underbrace{\langle 1|\hat{\rho}|1\rangle}_{1/2} \langle 1|$$

$S(\hat{\rho}') \Rightarrow$ basis $\{|0\rangle, |1\rangle\}$

$$\hat{\rho}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow S(\hat{\rho}') = \ln 2 > S(\hat{\rho}) = 0$$

Most general dynamics for $\hat{\rho}$ under an environment:

Choi-Kraus repr. theorem:

$$\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^+ \qquad \Gamma \leq N^2$$

$N =$ dim. of Hilbert space

Oper: $\hat{M}_\gamma$: Kraus operators

satisfy: $\sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \hat{I}$ identity

previous examples:

i) closed sys: $\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^+$

$$\Gamma = 1 \qquad \hat{M}_1 = \hat{U}$$

$$\sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \hat{U}^+ \hat{U} = \hat{I}$$

ii) measurement: $\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^+$

$$\Gamma = N$$

$$\hat{M}_\gamma \Rightarrow \hat{P}_a = |a\rangle\langle a|$$

$$\sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \sum_a |a\rangle\langle a|a\rangle\langle a|$$

$$= \sum_a |a\rangle\langle a| = \hat{I}$$

note: in general $\hat{M}_\gamma^+ \neq \hat{M}_\gamma$

$\hat{M}_\gamma$ not necessarily unitary

$\hat{M}_\gamma \propto |1\rangle\langle 0|$ as an example

Proof: how to transform from a valid $\hat{\rho}$ to another valid $\hat{\rho}'$

valid: $\hat{\rho}^{\dagger} = \hat{\rho}$ $\quad tr(\hat{\rho}) = 1$ $\quad \langle i | \hat{\rho} | i \rangle \geq 0$
in any basis

also demand transform be linear:

example:
2 ensembles (pure)
A, B

$$\hat{\rho}_A = |\psi_1\rangle\langle\psi_1| \xrightarrow{\text{time evol.}} \hat{\rho}_A'$$
$$\hat{\rho}_B = |\psi_2\rangle\langle\psi_2| \rightsquigarrow \hat{\rho}_B'$$

third ensemble:
frac. f of $|\psi_1\rangle$
" $1-f$ of $|\psi_2\rangle$

$$\hat{\rho}_C = f|\psi_1\rangle\langle\psi_1| + (1-f)|\psi_2\rangle\langle\psi_2|$$
$$= f\hat{\rho}_A + (1-f)\hat{\rho}_B$$
$$\xrightarrow{\text{time evol.}} \hat{\rho}_C' = f\hat{\rho}_A' + (1-f)\hat{\rho}_B'$$

time evolution preserves linear combo's of density operators