proof of Choi-Kraus repr. theorem:

$$\hat{\rho} = \sum_{ij} \rho_{ij} |i\rangle\langle j| \qquad \rho_{ij} = \langle i|\hat{\rho}|j\rangle$$

$\{|i\rangle\}$: N-dim basis of our Hilbert space

$$\hat{\rho}' = \sum_{\ell k} \rho'_{\ell k} |\ell\rangle\langle k| \qquad \rho'_{\ell k} = \langle \ell|\hat{\rho}'|k\rangle$$

linear transform: $\hat{\rho} \Rightarrow \hat{\rho}'$ w/ tensor S

$$\rho'_{\ell k} = \sum_{ij} \underbrace{S_{\ell i;kj}}_{\substack{\text{4 index tensor} \\ \text{each element = number}}} \rho_{ij}$$

RHS is a linear combo of ρ_{ij} elements

multiply both sides by $|\ell\rangle\langle k|$ + sum over ℓ, k:

$$\hat{\rho}' = \sum_{\ell k} \rho'_{\ell k} |\ell\rangle\langle k| = \sum_{ij\ell k} S_{\ell i;kj} \overbrace{\langle i|\hat{\rho}|j\rangle}^{\rho_{ij}} |\ell\rangle\langle k|$$

$$= \sum_{ij\ell k} S_{\underbrace{\ell i}_{\alpha};\underbrace{kj}_{\beta}} \overbrace{|\ell\rangle\langle i|}^{\hat{\tau}_\alpha} \hat{\rho} \overbrace{|j\rangle\langle k|}^{\hat{\tau}_\beta^+}$$

$\alpha = (\ell, i)$ $[N^2$ dim.$]$ $\beta = (k, j)$

$\ell = 1 \cdots N$, $i = 1 \cdots N$

$(1,1) \Rightarrow \alpha = 1$ .... $(N,N) \Rightarrow \alpha = N^2$

$(1,2) \Rightarrow \alpha = 2$

$\hat{\tau}_\alpha = |\ell\rangle\langle i| \qquad \hat{\tau}_\alpha^+ = |i\rangle\langle \ell|$

$$\hat{\rho}' = \sum_{\alpha\beta} S_{\alpha\beta} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger$$

$S_{\alpha\beta}$ as an element of a matrix S
[$N^2 \times N^2$ dim.]

general linear transf:

(1) $$\hat{\rho}' = \sum_{\alpha\beta} S_{\alpha\beta} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger$$

$\alpha = (l, i)$
$\beta = (k, j)$
$\tau_\alpha \equiv |l\rangle\langle i|$

enforce $\hat{\rho}'$ is a valid dens. operator:

i) $\hat{\rho}'^\dagger = \hat{\rho}'$: $$\hat{\rho}'^\dagger = \sum_{\alpha\beta} S_{\alpha\beta}^* \hat{\tau}_\beta \hat{\rho} \hat{\tau}_\alpha^\dagger$$

transform: $\alpha \to \beta$ $\beta \to \alpha$

$$= \sum_{\alpha\beta} S^*_{\beta\alpha} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}^+_\beta$$

$$\Rightarrow S^*_{\beta\alpha} = S_{\alpha\beta}$$ S matrix is Hermitian $[N^2 \times N^2]$

there must exist a matrix U that diag S:

$$U^+ S \underset{\substack{\downarrow \\ \text{cols: e-vecs of } S}}{U} = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$\hookrightarrow$ e-vals of S : λ_γ

$$U^+ U = U U^+ = I$$

$$S = U \Lambda U^+$$

$$S_{\alpha\beta} = \sum_{\gamma,\mu} U_{\alpha\gamma} \underbrace{\Lambda_{\gamma\mu}}_{\lambda_\gamma \delta_{\gamma\mu}} (U^+)_{\mu\beta}$$

$\Downarrow$ $U^+_{\gamma\beta} = U^*_{\beta\gamma}$

$$= \sum_\gamma U_{\alpha\gamma} \lambda_\gamma U^*_{\beta\gamma}$$

plug into Eq. (1):

$$\hat{\rho}' = \sum_{\alpha\beta\gamma} \lambda_\gamma U_{\alpha\gamma} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}^+_\beta U^*_{\beta\gamma}$$

define operators (Krans op's)

$$\hat{M}_\gamma = \sum_\alpha U_{\alpha\gamma} \hat{\tau}_\alpha \sqrt{|\lambda_\gamma|}$$

$$\hat{M}^+_\gamma = \sum_\alpha U^*_{\alpha\gamma} \hat{\tau}^+_\alpha \sqrt{|\lambda_\gamma|}$$

$$= \sum_\beta U^*_{\beta\gamma} \hat{T}^\dagger_\beta \sqrt{|\lambda_\gamma|} \qquad \alpha \to \beta$$

$$\Rightarrow \quad \hat{\rho}' = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}^\dagger_\gamma \qquad \epsilon_\gamma = \frac{\lambda_\gamma}{|\lambda_\gamma|} = \pm 1$$

$$= \mathrm{sign}(\lambda_\gamma)$$

ii) $\mathrm{tr}(\hat{\rho}') = 1$

$$= \sum_\gamma \epsilon_\gamma \, \mathrm{tr}(\hat{M}_\gamma \hat{\rho} \hat{M}^\dagger_\gamma)$$

$\mathrm{tr}(\hat{A}\hat{B}\hat{C})$

$= \mathrm{tr}(\hat{C}\hat{A}\hat{B})$

$$= \sum_\gamma \epsilon_\gamma \, \mathrm{tr}(\hat{M}^\dagger_\gamma \hat{M}_\gamma \hat{\rho})$$

$\mathrm{tr}(\hat{A} + \hat{B})$

$= \mathrm{tr}(\hat{A}) + \mathrm{tr}(\hat{B})$

$$= \mathrm{tr}\Bigg(\underbrace{\Big[\sum_\gamma \epsilon_\gamma \hat{M}^\dagger_\gamma \hat{M}_\gamma \Big]}_{\hat{A}} \hat{\rho} \Bigg)$$

$\hat{A}$ = some oper.

$\hat{A}^\dagger = \hat{A}$ Hermitian

$$1 = \mathrm{tr}(\hat{A}\hat{\rho})$$

eval. in a basis where $\hat{A}$ is diag: $\{|m\rangle\}$

$$1 = \sum_m \langle m | \hat{A} \hat{\rho} | m \rangle$$

$$= \sum_{m,n} \underbrace{\langle m | \hat{A} | n \rangle}_{A_{mm}\delta_{mn}} \langle n | \hat{\rho} | m \rangle$$

$$1 = \sum_m A_{mm} \rho_{mm}$$

but we also know $tr(\hat{\rho}) = \sum_m \rho_{mm} = 1$

. only way both true for any $\hat{\rho}$:

$$A_{mm} = 1 \quad \text{for all } m$$

$$\hat{A} = \hat{I} = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma$$