$$\hat{\rho}' = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \qquad \epsilon_\gamma = \pm 1$$

$$\sum_\gamma \epsilon_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{I}$$

iii) in any basis $\langle i|\hat{\rho}'|i\rangle \geq 0$

$$\langle i|\hat{\rho}'|i\rangle = \sum_\gamma \epsilon_\gamma \langle i|\hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger|i\rangle \geq 0$$

true for any $\hat{\rho} \Rightarrow$ choose $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\Rightarrow \langle i|\hat{\rho}'|i\rangle = \sum_\gamma \epsilon_\gamma \langle i|\hat{M}_\gamma|\psi\rangle\langle\psi|\hat{M}_\gamma^\dagger|i\rangle$$

$$= \sum_\gamma \epsilon_\gamma |\langle i|\hat{M}_\gamma|\psi\rangle|^2 \geq 0$$

since true for any $|\psi\rangle \Rightarrow \epsilon_\gamma = +1$ for all γ

Choi-Kraus:
(CK)

$$\boxed{\hat{\rho}' = \sum_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \quad \text{where} \quad \sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{I}}$$

note: set of $\hat{M}_\gamma$ operators not unique

Goal: qu. master equation

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \sim\sim\sim$$

CK applies at every time step:

$$\hat{\rho}(t+\delta t) = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \, \hat{\rho}(t) \, \hat{M}_\gamma^{+} \quad (1)$$

$\Gamma \leq N^2$

N = dim. Hilbert space

assumption: dynamics is Markovian

$$\Rightarrow \hat{\rho}(t+\delta t) = \hat{\rho}(t) + \sim\sim \delta t$$

$$\Rightarrow \underbrace{\frac{\hat{\rho}(t+\delta t) - \hat{\rho}(t)}{\delta t}}_{\frac{\partial}{\partial t}\hat{\rho}(t)} = \sim\sim$$

easy case: no interactions w/ env.

$$\hat{\rho}(t+\delta t) = \hat{U}_s \, \hat{\rho}(t) \, \hat{U}_s^{+}$$

$\Gamma = 1$

$\hat{U}_s = e^{-\frac{i\hat{\mathcal{H}}_s \delta t}{\hbar}}$

$\hat{U}_s^{+}\hat{U}_s = \hat{I}$

sys. Hamiltonian: $\hat{\mathcal{H}}_s$ time-ind.

$\hat{M}_1 = \hat{U}_s \qquad \hat{M}_\gamma = 0 \quad \gamma > 1$

$$\hat{\rho}(t+\delta t) = \left(\hat{I} - \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \dots\right) \hat{\rho}(t) \left(\hat{I} + \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \dots\right)$$

$$= \hat{\rho}(t) - \frac{i}{\hbar}\left[\hat{\mathcal{H}}_s, \hat{\rho}(t)\right] \delta t + \dots$$

$$\Rightarrow \boxed{\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{\mathcal{H}}_s, \hat{\rho}\right]}$$

qu. master eq. for isolated sys. (von Neumann equ.)

turn on interactions;

CK matrices: $\hat{M}_1 = \hat{I} - \frac{i\hat{H}_s \delta t}{\hbar} + \underbrace{\hat{K}\,\delta t}_{\text{correction due to env. interactions}}$

$[N^2-1]$ $\quad \gamma > 1: \quad \hat{M}_\gamma = \sqrt{\delta t}\,\hat{L}_\gamma$ for some oper's $\hat{L}_\gamma$

$\hat{L}_\gamma, \hat{K}$ describe interactions:

to turn off inter. $\hat{L}_\gamma, \hat{K} \to 0$

demand:
$$\hat{I} = \sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma$$
$$= \hat{I} + \delta t \underbrace{\left[2\hat{K} + \sum_{\gamma>1} \hat{L}_\gamma^\dagger \hat{L}_\gamma\right]}_{=0} + \cdots$$

$$\hat{K} = -\frac{1}{2}\sum_{\gamma>1} \hat{L}_\gamma^\dagger \hat{L}_\gamma$$

previous:
$$\hat{I} = \hat{M}_1^\dagger \hat{M}_1$$
$$= \left(\hat{I} + \frac{i\hat{H}_s\delta t}{\hbar}\right)\left(\hat{I} - \frac{i\hat{H}_s\delta t}{\hbar}\right)$$
$$= \hat{I} + \underbrace{\cdots\cdots}_{0} + O(\delta t^2)$$

plug $\hat{M}_1, \hat{M}_2, \hat{M}_3$, etc. into Eq. (1):

algebra $\Rightarrow$

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{\mathcal{H}}_s, \hat{\rho}] + \sum_{\gamma>1}\left(\hat{L}_\gamma \hat{\rho} \hat{L}_\gamma^\dagger - \frac{1}{2}\hat{L}_\gamma^\dagger \hat{L}_\gamma \hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_\gamma^\dagger \hat{L}_\gamma\right)}$$

qu. master eq. w/ env. interactions

Lindblad eq. (1976)

up to N^2-1 terms in sum

next step: intuitive explanation for $\hat{L}_\gamma$

can prove Lindblad eq. is invariant under:
(structure preserved)

i) $\hat{L}_\gamma \to \hat{L}'_\gamma = \sum_\alpha U_{\gamma\alpha} \hat{L}_\alpha$ where $U_{\gamma\alpha}$ are components of any unitary $(N^2-1)\times(N^2-1)$ matrix

ii) $\hat{L}_\gamma \to \hat{L}'_\gamma = \hat{L}_\gamma + c_\gamma$ $\quad c_\gamma =$ complex const.

$$\hat{\mathcal{H}}_s \to \hat{\mathcal{H}}'_s = \hat{\mathcal{H}}_s \underbrace{- \frac{i\hbar}{2}\sum_{\gamma>1}\left(c_\gamma^* \hat{L}_\gamma - c_\gamma \hat{L}_\gamma^\dagger\right)}_{\text{Hermitian}}$$

using these freedoms, we choose a set of $\hat{L}_\gamma$ w/ following properties:

$\mathrm{tr}(\hat{L}_\gamma) = 0$ for $\gamma > 1$ [using ii choosing c_γ]

$\mathrm{tr}(\hat{L}_\gamma \hat{L}_\alpha^+) = a_\gamma \delta_{\gamma\alpha}$ where $a_\gamma \geq 0$ [using i, choosing U]

Hilbert-Schmidt inner prod. for oper's $\hat{A}, \hat{B}$ $\Rightarrow \mathrm{tr}(\hat{A}\hat{B}^+)$

Write: $\hat{L}_\gamma = \sqrt{a_\gamma}\,\hat{\Lambda}_\gamma \Rightarrow \mathrm{tr}(\hat{\Lambda}_\gamma \hat{\Lambda}_\alpha^+) = \delta_{\gamma\alpha}$

$\mathrm{tr}(\hat{\Lambda}_\gamma) = 0$

unique set of $\hat{\Lambda}_\gamma$ satisfying these:

Hilbert space basis $\{|i\rangle\}$ N-dim.

"operator basis": two types of $\hat{\Lambda}_\gamma$ operators

i) $\hat{\Lambda}_\gamma = |i\rangle\langle j|$ $i \neq j$

"jump" $j \to i$ $\hat{\Lambda}_\gamma |j\rangle = |i\rangle$

$N(N-1)$ distinct jump oper.

ii) $\hat{\Lambda}_\gamma = \dfrac{\hat{\mathbb{1}} - 2|i\rangle\langle i|}{\sqrt{N}}$ $i = 2, 3, \ldots, N$

total $N^2 - 1$ oper.

$N-1$ operators: "dephasing" operators

$\hat{\Lambda}_\gamma \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) =$ result has

diff. phase b/t
$|0\rangle$ & $|1\rangle$