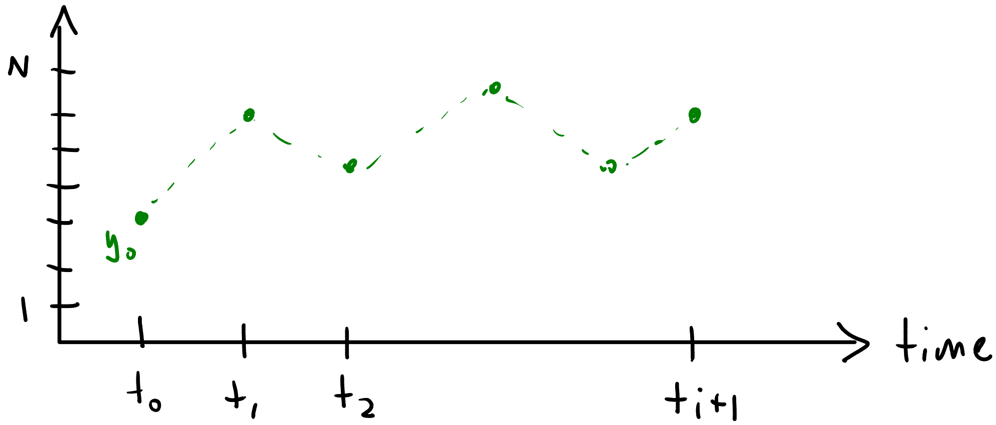
Recall: trajectory $\nu = (y_0, y_1, \dots)$



$\mathcal{P}(\nu)$ = prob. to observe ν in ensemble

key assumption: Markovian dynamics

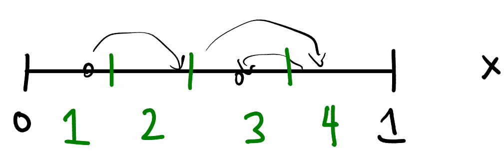
$$\mathcal{P}(y_{i+1} \mid y_0, y_1, \dots, y_i)$$

prob. of y_{i+1} given past history $y_0, \dots, y_i$

$$= \mathcal{P}(y_{i+1} \mid y_i)$$

only depends on immediate past state y_i

example from prob. 1 in PS #1



run simulations: traj.

$(1, 1, 2, 1, 3)$

$(1, 2, 3, 4, 2)$

$\circ \; \circ \; \circ$

$(y_0, y_1, y_2, y_3, y_4)$

$$\mathcal{P}(y_4 = 3 \mid y_2 = 4,\ y_3 = 1) \overset{?}{=} \mathcal{P}(y_4 = 3 \mid y_3 = 1)$$

check

$$\frac{\#\text{ traj of form } (*,*,4,1,3)}{\#\text{ traj. of form } (*,*,4,1,*)} \stackrel{?}{=} \frac{(*,*,*,1,3)}{(*,*,*,1,*)}$$

$$\left(\frac{\hat{H}(t)}{i\hbar} + \hat{I}\right)\Delta t \, |\psi(t)\rangle$$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

$$||$$

$$|\psi(t+\Delta t)\rangle = \left(|\psi(t)\rangle + \frac{\hat{H}(t)\, |\psi(t)\rangle}{i\hbar}\right) \Delta t$$

Why is Markovianity useful?

1) makes calculating $\mathcal{P}(\nu)$ easier:

$$\mathcal{P}(A|B) = \frac{\mathcal{P}(A,B)}{\mathcal{P}(B)}$$

LHS:

$$\frac{\mathcal{P}(y_0, y_1, \ldots, y_{i+1})}{\mathcal{P}(y_0, y_1, \ldots, y_i)} \underset{\text{Markov}}{=} \mathcal{P}(y_{i+1} | y_i)$$

$$\underbrace{\mathcal{P}(y_0, y_1, \ldots, y_{i+1})}_{\mathcal{P}(\nu)} = \mathcal{P}(y_{i+1}|y_i)\, \mathcal{P}(y_0, \ldots, y_i)$$

$= \text{recursion}$

$$= \mathcal{P}(y_{i+1}|y_i)\, \mathcal{P}(y_i | y_{i-1}) \cdots$$

$$= \left[\prod_{j=0}^{i} \mathcal{P}(y_{j+1} | y_j)\right] \underset{\substack{\downarrow \\ \text{initial dist.}}}{\mathcal{P}(y_0)}$$

$y_{j+1} = 1, \ldots, N \qquad y_j = 1, \ldots, N \qquad N = \#$ states

$$\mathcal{P}(y_{j+1} = n \mid y_j = m) \equiv W_{nm}(t_j)$$

"transition probability"
given m prob. of
going to n

$W(t_j) = N \times N$ matrix
of probabilities
(all real, b/t 0 and 1)
[not necessarily symmetric]

$$\Rightarrow \boxed{\underset{\substack{\parallel \\ (n_0, n_1, n_2, \ldots, n_{i+1})}}{\mathcal{P}(\nu)} = W_{n_{i+1} n_i}(t_i)\, W_{n_i n_{i-1}}(t_{i-1}) \cdots W_{n_1 n_0}(t_0)\, \mathcal{P}(n_0)}$$

$\mathcal{P}(A) = \sum_B \mathcal{P}(A, B)$

2) we can find prob. of being in
some state at current time:

$$\mathcal{P}(y_{i+1}) = \sum_{y_0=1}^{N} \cdots \sum_{y_i=1}^{N} \mathcal{P}(y_0, y_1, \ldots, y_{i+1})$$

$$\overset{\text{Markov}}{=} \sum_{y_0} \cdots \sum_{y_{i-1}} \sum_{y_i} \mathcal{P}(y_{i+1} \mid y_i)\, \mathcal{P}(y_0, \ldots, y_i)$$

$$= \sum_{y_i} \mathcal{P}(y_{i+1} \mid y_i)\, \mathcal{P}(y_i)$$

rewrite in matrix-vector form:

$\mathcal{P}(y_i = m) = p_m(t_i)$

$\mathcal{P}(y_{i+1} = n) = p_n(t_{i+1})$

elements of

$\vec{p}(t_i)$ $\vec{p}(t_{i+1})$

$$p_n(t_{i+1}) = \sum_{m=1}^{N} W_{nm}(t_i)\, p_m(t_i)$$

$$\Rightarrow \boxed{\vec{p}(t_{i+1}) = W(t_i)\, \vec{p}(t_i)}$$

discrete-state discrete-time master equation

$$\vec{p}(t_{i+1}) = W(t_i) \cdots W(t_1)\, W(t_0)\, \vec{p}(t_0)$$