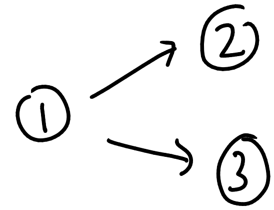
$$\vec{p}(t_{i+1}) = W\,\vec{p}(t_i)$$

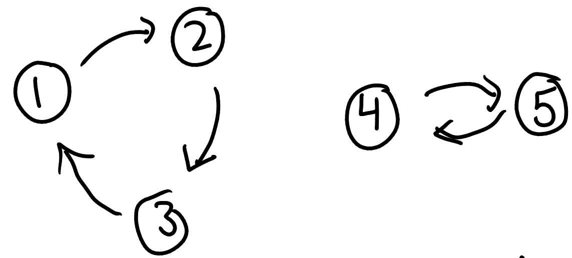
$$\vec{p}^{\,s} = W\vec{p}^{\,s}$$

example #1:



∞ # of $\vec{p}^{\,s}$

example #2:



$$W = \begin{array}{cc} & \begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \end{array} \ \text{start} \\ \text{end}\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left(\begin{array}{c|c} \quad & 0 \\ \hline 0 & \quad \end{array}\right) \end{array}$$

$$\vec{p}_A^{\,s} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix} \qquad \vec{p}_B^{\,s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{pmatrix}$$

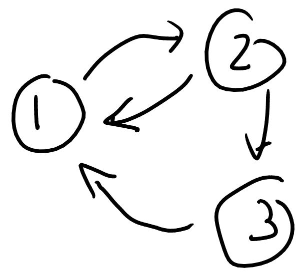
$$\vec{p}_C^{\,s} = \alpha\,\vec{p}_A^{\,s} + (1-\alpha)\,\vec{p}_B^{\,s} \qquad 0 \le \alpha \le 1$$

Restrict focus to systems w/ unique $\vec{p}^{\,s}$:

- always consider connected graphs (avoiding example #2)
- demand our graph is strongly connected:

if start at any state we can reach any other state following arrows

strongly conn. example



also known: ergodic graph

as $t \to \infty$ you will visit all states

- final condition (will prove later for classical & quantum sys):

microscopic reversibility (MR)

if $W_{ij} \neq 0 \Rightarrow W_{ji} \neq 0$

$j \to i$ arrow exists; $i \to j$ arrow exists

(either no arrow or double arrow b/t any two states)

Categories:

- all graphs
  - connected
    - ergodic
      - $\boxed{\text{MR}}$
      - non-MR
    - non-ergodic
  - disconnected

Sketch out next steps:

- prove MR graphs have unique $\vec{p}^{\,s}$

- explore why physics is MR

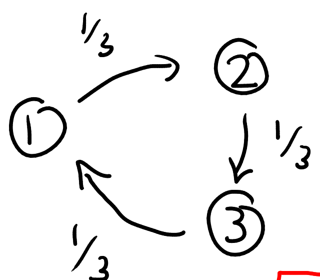
→ to do this we need to intro "mean hitting time" of a graph

$\underbrace{\text{mean hitting time}}_{\text{MHT}}$

MHT = avg. # of time steps δt to get to state j given start at i ("first passage times")

$\equiv h_{ji}$ ($< \infty$ for ergodic graphs)

example:



length 2

trajectory: 1 1 2 2 3 3 3 1 2 3

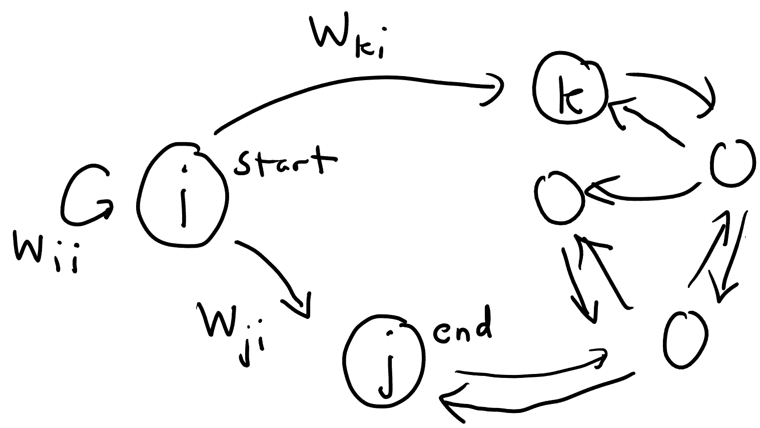
len. 1 len. 1

h_{21} = avg. of all trip lengths from 1 to 2

h_{11} = " " " 1 to 1

N^2 quantities h_{ij} to calculate

$\Rightarrow$ N^2 equations?



$$h_{ji} = W_{ji} \cdot \underset{\substack{\uparrow \\ \text{trip length}}}{1} + \sum_{k \neq j} W_{ki} (1 + h_{jk})$$

$$h_{ji} = \underbrace{W_{ji} + \sum_{k \neq j} W_{ki}}_{= \sum_{k} W_{ki} = 1} + \sum_{k \neq j} W_{ki} h_{jk}$$

$$\Rightarrow \boxed{h_{ji} = 1 + \sum_{k \neq j} h_{jk} W_{ki}}$$

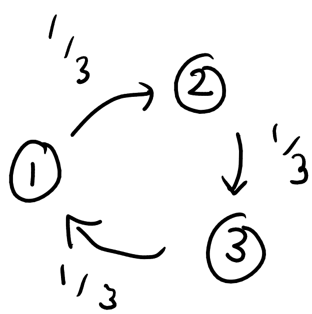
$i = 1, \ldots, N$

$j = 1, \ldots, N$

Eq. (*)

N^2 equations for N^2 unknowns h_{ji}

example:



$$W = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} \end{array}$$

$i=1, j=1$: $$h_{11} = 1 + \sum_{k \neq 1} h_{1k} W_{k1} = 1 + h_{12} \overbrace{W_{21}}^{1/3}$$

$i=2, j=1$: $$h_{12} = 1 + h_{12} W_{22} + h_{13} W_{32}$$

$i=3, j=1$: $$h_{13} = 1 + h_{13} W_{33}$$

$\Rightarrow \quad h_{13} = 3 \qquad h_{12} = 6 \qquad h_{11} = 3$

Why is this useful?

proof of uniqueness of $\vec{p}^{\,s}$ in an ergodic graph:

1) given a W, choose any $\vec{p}^{\,s}$ (at least one exists)

2) multiply both sides of Eq. $*$ by p_i^s:

$$p_i^s h_{ji} = p_i^s \left(1 + \sum_{k \neq j} h_{jk} W_{ki}\right)$$

3) sum both sides over i:

$$\sum_i p_i^s h_{ji} = 1 + \sum_i p_i^s \sum_{k \neq j} h_{jk} W_{ki}$$

$$p_j^s h_{jj} + \sum_{i \neq j} p_i^s h_{ji} = 1 + \sum_{k \neq j} h_{jk} \underbrace{\sum_i W_{ki} p_i^s}_{p_k^s}$$

b/c $W\vec{p}^{\,s} = \vec{p}^{\,s}$

$$p_j^s h_{jj} + \sum_{i \neq j} p_i^s h_{ji} = 1 + \sum_{k \neq j} p_k^s h_{jk}$$

same sum
diff. dummy var.

$$p_j^s h_{jj} = 1 \implies p_j^s = \frac{1}{h_{jj}} > 0$$

b/c $h_{jj} < \infty$ in an ergodic net

$\implies$ unique solution for $\vec{p}^s$