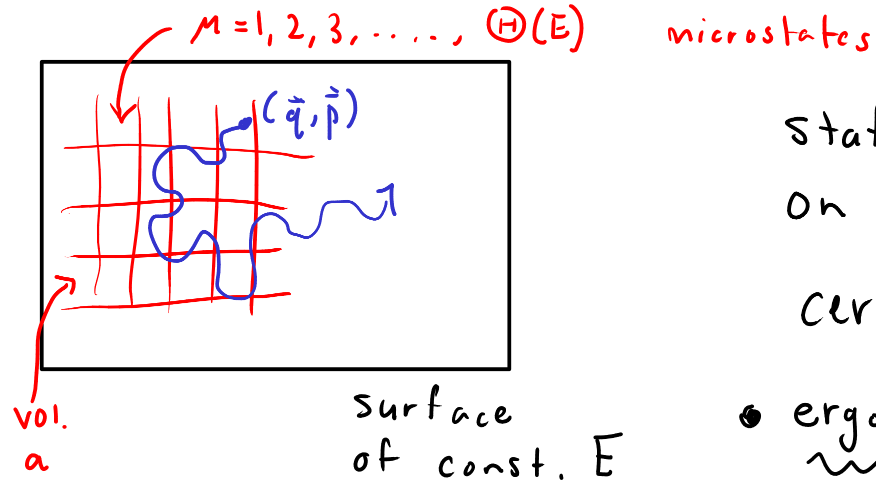


surface of const. E in $(\vec{q}, \vec{p})$ space

stat. mech focuses on systems w/ certain properties:

- ergodicity: for any traj as $t \to \infty$ every microstate μ is visited (no matter how small a is)

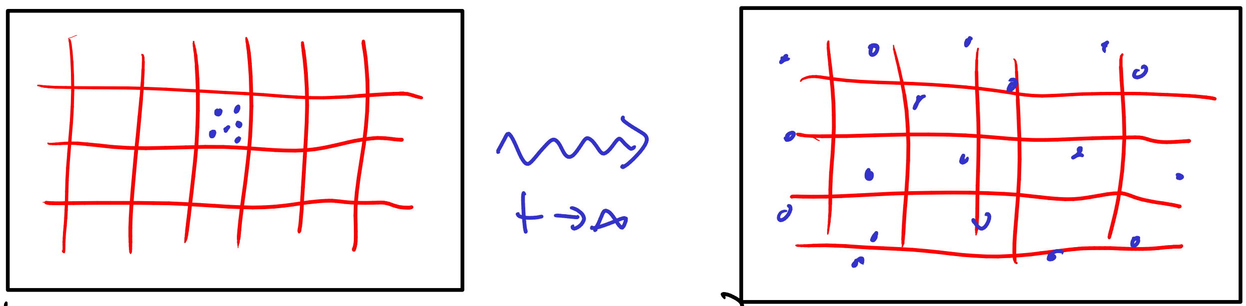
note: in systems w/ non-trivial conserved quantities besides E this may be violated

2d harm. oscill: $E = E_x + E_y$

$$\left.\begin{aligned} E_x &= \frac{1}{2} p_x^2 + \frac{1}{2} x^2 \\ E_y &= \frac{1}{2} p_y^2 + \frac{1}{2} y^2 \end{aligned}\right\} \text{separately conserved}$$

- mixing: if we initiate a group of traj. at $t=0$ in same microstate μ
  $\Rightarrow$ traj. spread out evenly to all microstates as $t \to \infty$

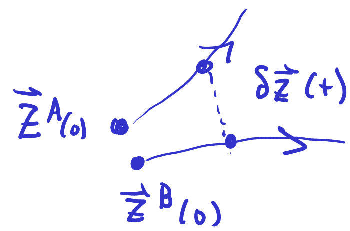
mixing $\subset$ ergodicity



How is this possible?

two traj. w/ slightly diff initial conditions:

$$\vec{z} = (\vec{q}, \vec{p})$$



$$\vec{z}^A(t) \qquad \vec{z}^B(t)$$

$$\delta\vec{z}(t) = \vec{z}^A(t) - \vec{z}^B(t)$$

chaos: $$|\delta\vec{z}(t)| \sim e^{\lambda t}|\delta\vec{z}(0)|$$

for small times t & small $|\delta\vec{z}(0)|$

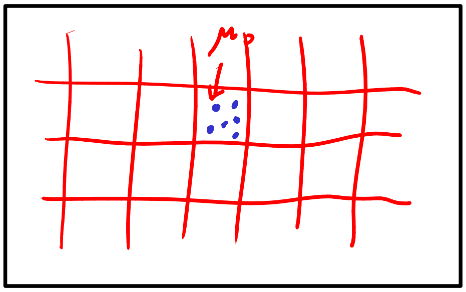
w/ $\lambda > 0$ Lyapunov exponent

this behavior is often associated w/ mixing but hard to prove rigorously

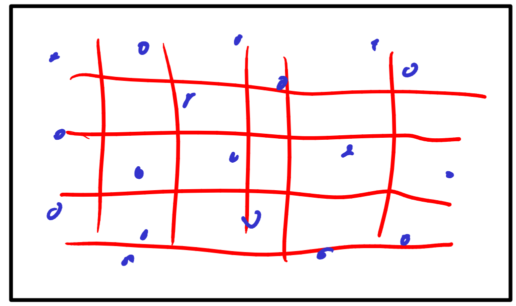
if mixing is true:

$P_\mu(t)$ = prob. to be in microstate μ at time t

$t = 0$ $\qquad\qquad\qquad\qquad t = \infty$ $\qquad\qquad \Theta(E) = \#$ boxes



$\rightsquigarrow$ $t \to \infty$



$$P_\mu(0) = \begin{cases} 1 & \text{if } \mu = \mu_0 \\ 0 & \text{if } \mu \neq \mu_0 \end{cases} \quad \underset{t\to\infty}{\rightsquigarrow} \quad P_\mu(t) = \frac{1}{\Theta(E)}$$

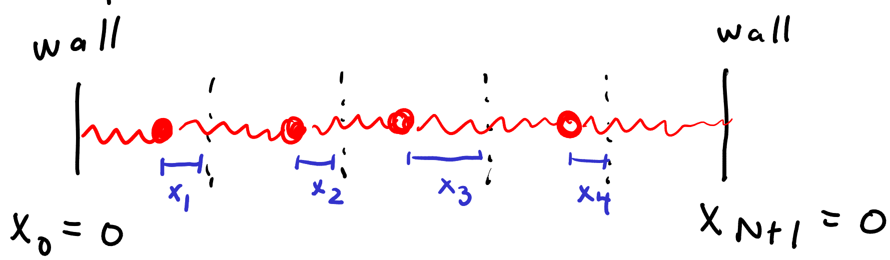
all microstates equally likely as $t \to \infty$:

microcanonical ensemble

First test: 1953-55 Fermi, Pasta, Ulam, Tsingou (FPUT)

first physics computer exper.

connected springs



N masses: $m = 1$

w/ spring constants $k = 1$

right spring force

left spring force

eq. of motion: $$\ddot{x}_n = \left[(x_{n+1} - x_n) - (x_n - x_{n-1})\right] \cdot \left(1 + \alpha (x_{n+1} - x_{n-1})\right)$$

when $\alpha \neq 0$ adds nonlinear contrib. to force

to get rid of conserved quantities

FPUT made springs non-Hookean

initially focus on $\alpha = 0$:

$$\mathcal{H}(\vec{x},\vec{p}) = \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + \frac{(x_n - x_{n-1})^2}{2} \right) + \frac{x_N^2}{2}$$

$$\vec{x} = (x_1, \ldots, x_N) \qquad p_n = \dot{x}_n$$
$$\vec{p} = (p_1, \ldots, p_n)$$

reminder: any funcs. $f(\vec{x},\vec{p})$ $g(\vec{x},\vec{p})$

Poisson bracket
$$\{f, g\}_{\vec{x},\vec{p}} \equiv \sum_{n=1}^{N} \left(\frac{\partial f}{\partial x_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial x_n} \right)$$

typically left out of notation

Can show: $\frac{d}{dt} f = \{f, \mathcal{H}\}$

choose
$$f = x_i \Rightarrow \dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$f = p_i \Rightarrow \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}$$
Hamilton's eqn's

$$\{x_i, x_j\} = 0 \qquad \{p_i, p_j\} = 0$$
$$\{x_i, p_j\} = \delta_{ij}$$

Canonical transf. (CT) of coordinates:

define $\vec{Q}(\vec{x},\vec{p})$, $\vec{P}(\vec{x},\vec{p})$

that preserve Poisson brackets:

$$\{f, g\}_{\vec{x}, \vec{p}} = \{f, g\}_{\vec{Q}, \vec{\mathcal{P}}}$$

$$\Rightarrow \quad \dot{Q}_n = \{Q_n, \mathcal{H}\}_{\vec{Q}, \vec{\mathcal{P}}} = \frac{\partial \mathcal{H}}{\partial \mathcal{P}_n}$$

$$\dot{\mathcal{P}}_n = \{\mathcal{P}_n, \mathcal{H}\}_{\vec{Q}, \vec{\mathcal{P}}} = -\frac{\partial \mathcal{H}}{\partial Q_n}$$

FPUT system example:

CT : new coordinates

"position" "momentum"

phase ϕ_i $\updownarrow$ x_i amplitude I_i $\updownarrow$ p_i

$$x_n = \sum_{k=1}^{N} \sqrt{\frac{2I_k}{(N+1)\omega_k}} \sin\left(\frac{nk\pi}{N+1}\right) \sin(\phi_k)$$

$$p_n = \sum_{k=1}^{N} \sqrt{\frac{2I_k\omega_k}{N+1}} \sin\left(\frac{nk\pi}{N+1}\right) \cos(\phi_k)$$

$$\omega_k = 2 \sin\left(\frac{k\pi}{2(N+1)}\right) \qquad k = 1, \ldots, N$$

Check: $\{x_i, p_j\}_{\vec{\phi}, \vec{I}} = \delta_{ij}$

after transf: $\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \underbrace{\overset{\text{const.}}{\omega_k} I_k}_{\text{energy of kth "normal" mode}}$