$$\mathcal{H}(\vec{\phi}, \vec{I}) = \sum_{k=1}^{N} \underbrace{\omega_k I_k}_{\text{kth normal mode energy}}$$

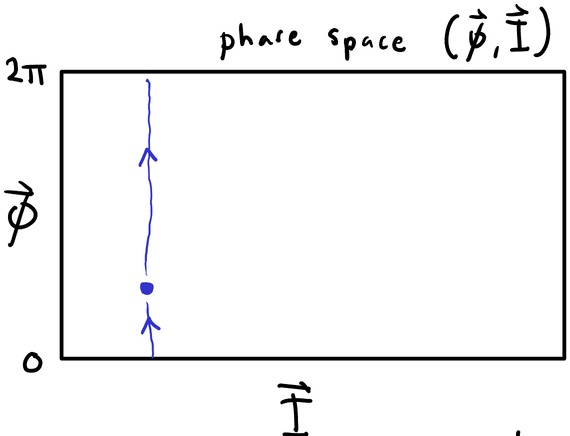
Hamilton's equ's in new coord

$$\dot{\phi}_i = \frac{\partial H}{\partial I_i} = \omega_i$$

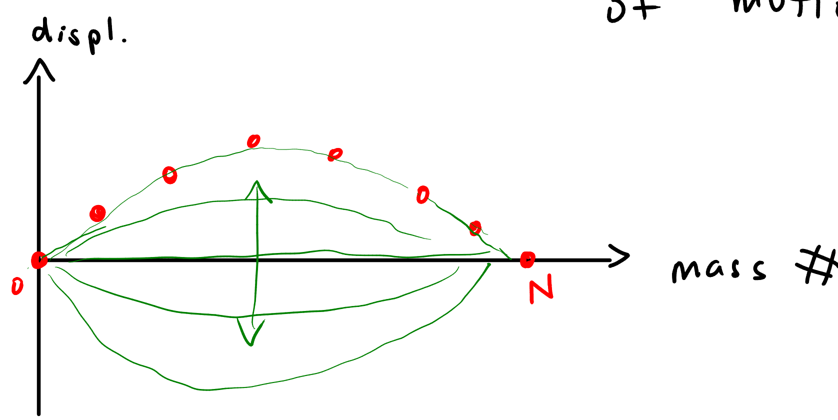
$$\Rightarrow \phi_i(t) = \omega_i t + \phi_i(0)$$

$$\dot{I}_i = -\frac{\partial H}{\partial \phi_i} = 0 \Rightarrow \text{all } I_i$$

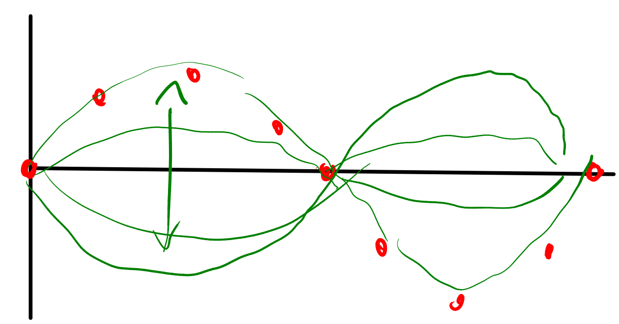
are constants of motion



$k=1$ mode



$k=2$ mode



What happens when $\alpha \neq 0$?

In same coords:

$$\mathcal{H}(\vec{\phi},\vec{I}) = \sum_{k=1}^{N} \omega_k I_k + \alpha U(\vec{\phi},\vec{I}) + \dots$$

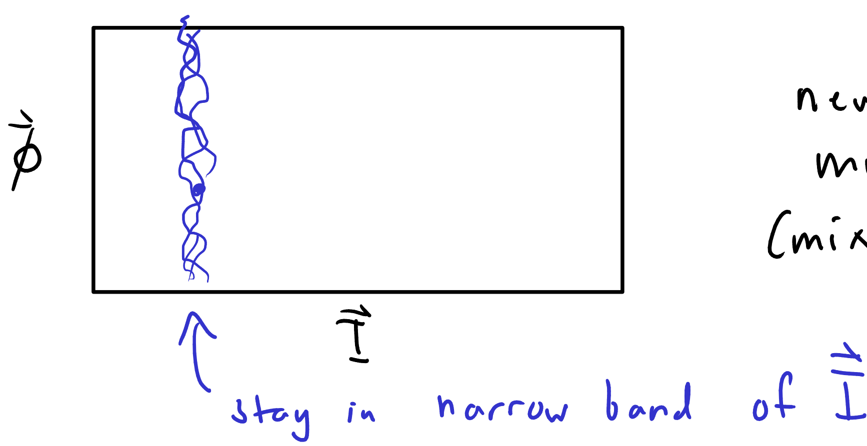
$\uparrow$ small α perturbation

higher order in α

idea: $\dot{I}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i} \neq 0 \Rightarrow I_i$ are no no longer constants of motion

$\Rightarrow$ all of phase space space should be explored?

Summary: $\alpha \neq 0$ small FPUT observed quasiperiodic behavior: you return close to (but not exactly) your initial conditions



never achieve microcanonical ensemble (mixing) expected by Fermi

Consider a more general class of systems $\mathcal{H}(\vec{x},\vec{p})$

$\vec{x} = (x_1, \dots, x_n)$

$\vec{p} = (p_1, \dots, p_n)$

2n-dim phase space

This system is integrable:

i) there n linearly indep. constants of motion: $F_k(\vec{x}, \vec{p}) \quad k = 1, \dots, n$

$$\Rightarrow \quad \frac{dF_k}{dt} = \{F_k, H\} = 0$$ b/c const. of motions

by convention: $F_1 = H$

ii) $\{F_k, F_\ell\} = 0$ for all k, ℓ

Consequence of integrability:

$\Rightarrow$ Liouville-Arnold theorem: if system is integrable, there exists a canonical transf. to "action-angle" coordinates:

$$(\vec{\phi}, \vec{I})$$

"angles" $(\phi_1, \dots, \phi_n)$

"actions" $(I_1, \dots, I_n)$

note: $\phi_i + 2\pi m \equiv \phi_i$ (m integer)

$$\mathcal{H} = \mathcal{H}(I)$$

$$\dot{I}_k = -\frac{\partial \mathcal{H}}{\partial \phi_k} = 0$$ all I_k are const. of motion

$$\phi_k(t) = \omega_k(I)t + \phi_k(0) \Leftarrow \dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial I_k}(\vec{I}) \equiv \omega_k(\vec{I})$$

const.

examples of integrable systems:

- all 1D problems w/ conserved energy
- n coupled harmonic springs
- central force problems
- two-body grav. problems
- gyroscopes + tops
- free particles confined on surfaces of ellipsoids

not integrable:

- three body problem (proven by Poincaré to be not integrable)
- chaotic systems
- dissipative systems