RECAP: coord. transf $\Rightarrow$ $(\vec{\phi}, \vec{I})$
angle action

integrable systems: $\mathcal{H} = \mathcal{H}(\vec{I})$

$$\dot{I}_k = -\frac{\partial \mathcal{H}}{\partial \phi_k} = 0$$

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial I_k}(\vec{I}) \equiv \omega_k(\vec{I})$$

integ. eq. of motion: $I_k =$ constant

$$\phi_k = \omega_k(\vec{I})\,t + \phi_k(0)$$

$$\phi_k + 2\pi = \phi_k$$

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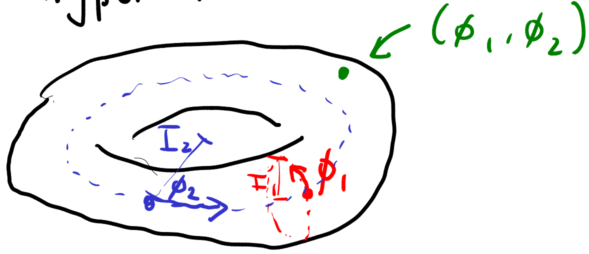
geometrical interpretation of integrable sys.

$\Rightarrow$ $\vec{\phi}(t) = (\phi_1, \phi_2, \ldots, \phi_n)$

coords on $T^n \simeq S^1 \times S^1 \times \cdots \times S^1$
hypertorus circle

n = 2:



Convention: I_1, I_2 two "radii" of torus

each choice of $\vec{I}$ consts of motion

$\Rightarrow$ one torus in a "foliation" of phase space

Classify tori:

1) <u>resonant tori:</u>

there exists a vector $\vec{v}$ of integers

$\vec{v} \neq 0 \qquad v_i \in \mathbb{Z}$

such that $\qquad \vec{v} \cdot \vec{\omega} = 0$

$$\vec{\omega} = (\omega_1(\vec{I}), \omega_2(\vec{I}), \ldots, \omega_n(\vec{I}))$$

special case: $\omega_i = \underset{\text{integer}}{Z_i} \cdot \text{constant}$

$\Rightarrow$ periodic, closed orbits on torus

2) non-resonant tori: no such $\vec{v}$ exists

$\Rightarrow$ never return to orig. position

$\Rightarrow$ traj. densely fill up torus

$\Rightarrow$ you will return arbitrarily close to initial position

$\Rightarrow$ quasiperiodicity (even if system is integrable)

What happens if you break integrability via a small perturbation?

$$\mathcal{H}(\vec{\phi}, \vec{I}) = \mathcal{H}_0(\vec{I}) + \alpha H_1(\vec{\phi}, \vec{I})$$

original integ. Hamiltonian

$\hookrightarrow$ small

Kolmogorov, Arnold, Moser (KAM) theorem:

1954 - 63

non-technical: for small α, donuts persist

$\Rightarrow$ traj. live on "deformed" donuts

KAM theorem:

conditions for our proof:

1) torus is strongly non-resonant

non-res: no $\vec{v} \neq 0$ exists where $\vec{v} \cdot \vec{\omega} = 0$

strongly non-res: for any $\vec{v} \neq 0$ of integers

$$\vec{v} \cdot \vec{\omega} \geq \frac{\epsilon}{|\vec{v}|^{\tau}}$$

for some real nums ϵ + τ $> n-1$

note: for small ϵ, most tori are strongly non-resonant

2) system is non-degenerate:

$$M_{ij} = \frac{\partial^2 H_0}{\partial I_i \partial I_j} = \frac{\partial \omega_i(\vec{I})}{\partial I_j}$$

$n \times n$ matrix

non-deg: $\det(M) \neq 0$

KAM statement: there exists $\delta > 0$ such that for perturb. $\alpha \leq \delta \epsilon^2$ all strongly non-resonant tori survive, + are only slightly deformed

$\Rightarrow$ we still get quasiperiod. trajectories

note: FPUT system is technically degenerate $\Rightarrow$ KAM does <u>not</u> apply

1970s: Nishida conj.: for low energies FPUT behaves like KAM

2005: Bob Rink proved conj.

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