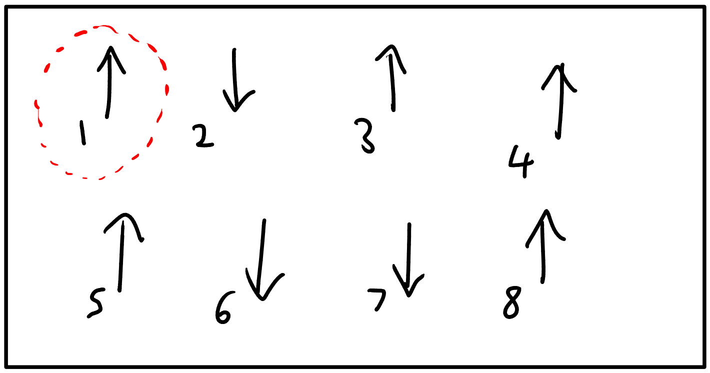
Example: "gas" of spins

spin 1 = "system" our focus



remaining M spins = "environment"

total:

k $\uparrow$ spins

$\underline{M+1-k}$ $\downarrow$ spins

$M+1$ spins

$k = 5$

$M + 1 = 8$

spin energies: 0 if $\downarrow$

ϵ if $\uparrow$

$$E_{tot} = k\epsilon$$

system states $N = 2$

$\Rightarrow$ spin 1 $\downarrow$ $E_1 = 0$ State 1

spin 1 $\uparrow$ $E_2 = \epsilon$ 2

environment: $E_1^{env} = E_{tot} - E_1 = k\epsilon$

$E_2^{env} = E_{tot} - E_2 = (k-1)\epsilon$

dynamics: at every time step δt one "collision" occurs: choose at random one $\downarrow$ and one $\uparrow$ spin & flip the pair

$\Rightarrow$ preserves $k \uparrow$ spins at all times: energy $E_{tot} = k\epsilon$ conserved

ergodic + mixing:

initial state

$k \uparrow$ spins

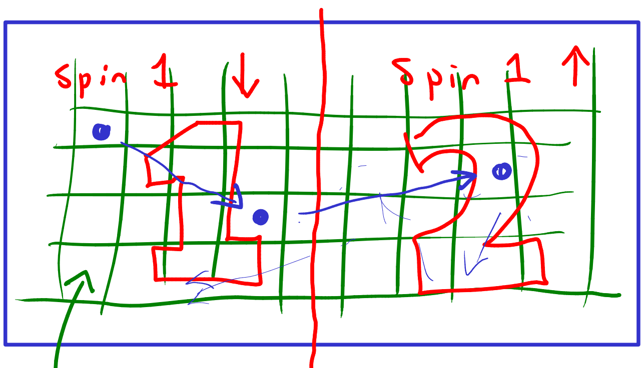
final state

$k \uparrow$ spins

1	2	3	....	M+1
$\uparrow$	$\downarrow$	$\downarrow$		$\uparrow$
$\downarrow$	$\downarrow$	$\uparrow$	....	$\uparrow$

can be done thru a seq. of "collisions"

1 1 0 0 $\qquad$ 0 0 1 1



each microstate: a spin config. of M+1 total spins

sys	env.		
1	2	3	......
$\downarrow$	$\uparrow$	$\downarrow$	......
$\downarrow$	$\downarrow$	$\uparrow$	...
$\uparrow$	$\downarrow$	$\downarrow$	...

How big are the macrostates: calculate

Θ_n = # envir. microstates when sys is in state n

Sys. state 1:

$$\begin{array}{c|c} 1 & 2 \;\cdots\; M+1 \\ \downarrow & \underbrace{\qquad\qquad\qquad}_{\text{arrangement of } k \text{ up spins}} \end{array}$$

$$\Theta_1 = \binom{M}{k} = \frac{M!}{(M-k)!\,k!}$$

$$= \frac{M\cdot(M-1)\cdots\cdots(M-k+1)}{k!}$$

Sys. state 2

$$\begin{array}{c|c} 1 & 2 \;\cdots\cdots\; M+1 \\ \uparrow & \underbrace{\qquad\qquad\qquad}_{k-1\ \uparrow \text{ spins arrangement}} \end{array}$$

$$\Theta_2 = \binom{M}{k-1} = \frac{M!}{(M-k+1)!\,(k-1)!}$$

transition matrix b/t macrostates

$$1 \underset{W_{12}}{\overset{W_{21}}{\rightleftarrows}} 2$$

W_{21} = prob. that in time step st spin 1 goes to ↑, given start in ↓

$$W_{21} = \frac{1}{\#\downarrow \text{spins}} = \frac{1}{M+1-k}$$

prob. of spin 1 $\downarrow$ participating in collision

$$W_{12} = \frac{1}{\#\uparrow \text{spins}}$$

prob. of spin 1 $\uparrow$
" " "

$$= \frac{1}{k}$$

$$\Theta_{tot} = \Theta_1 + \Theta_2 \qquad p_1^s = \frac{\Theta_1}{\Theta_{tot}} \qquad p_2^s = \frac{\Theta_2}{\Theta_{tot}}$$

check LDB: $W_{12}\, p_2^s = W_{21}\, p_1^s$

$$\frac{W_{12}}{W_{21}} = \frac{p_1^s}{p_2^s} = \frac{\Theta_1}{\Theta_2} = \frac{(M-k+1)!\,(k-1)!}{(M-k)!\,k!}$$

$$= \frac{M-k+1}{k} \quad \checkmark$$

derive: temperature

some func.

$E_{tot} = k\epsilon$

start: $\Theta_n = \Theta(E_{tot} - E_n)$

energy in env. in state n

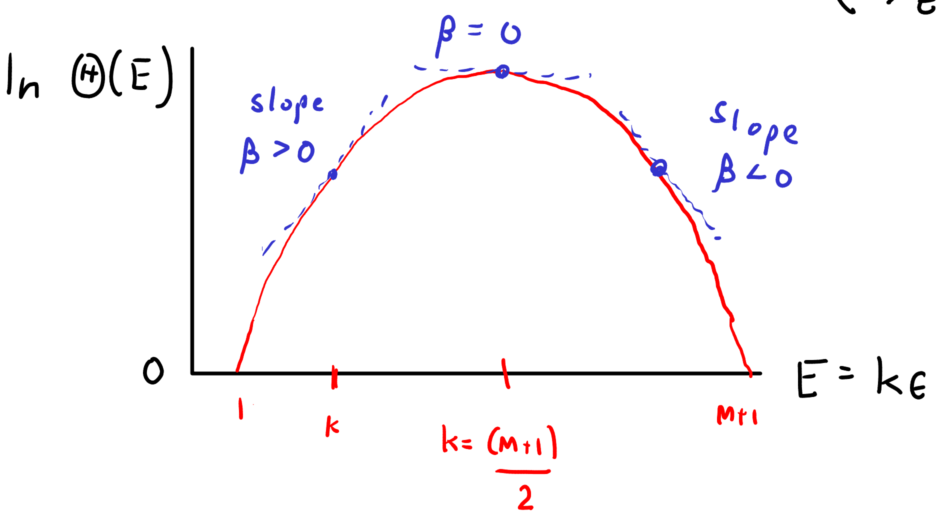
$$\Theta_1 = \binom{M}{k} = \binom{M}{E_{tot}/\epsilon}$$

$$\Theta_2 = \binom{M}{k-1} = \binom{M}{\frac{(E_{tot}-\epsilon)}{\epsilon}}$$

$E_{tot} - E_1 = E_{tot}$ b/c $E_1 = 0$

$E_{tot} - E_2 = E_{tot} - \epsilon$ b/c $E_2 = \epsilon$

$$\Rightarrow \quad \Theta(E) = \binom{M}{E/\epsilon}$$ universal func.



$$\beta = \frac{1}{k_B T} = \left.\frac{\partial \ln \Theta}{\partial E}\right|_{E_{tot}}$$