Linear thermodynamics:

focus on one time step δt:

$$\mu_0 = (n_0, n_1)$$

state n_0: $p^s_{n_0}$

" n_1: $p^s_{n_1}$

we are in a stationary state (not necessarily equil, $\dot{I}$ small but nonzero)

NESS: $\dot{I} = -\frac{1}{T}\dot{W}$

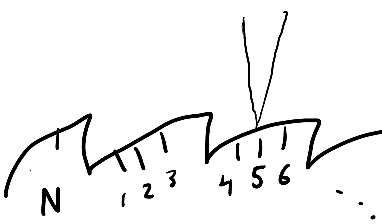
$$\langle I(\mu_0)\rangle = -\frac{1}{T}\langle W(\mu_0)\rangle$$

assume form for work: $W(\mu_0) = \omega_{n_1 n_0}$

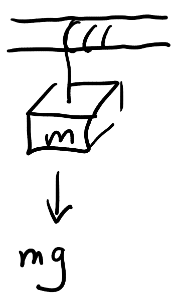
$$= -f\,\delta x_{n_1 n_0}$$

↑ "force" — some physical quantity we control

↑ "distance" — change in phys. observ.



$N \rightleftarrows 1 \rightleftarrows 2 \rightleftarrows 3 \cdots$



$f = mg$

$$\delta x_{n_1 n_0} = \begin{cases} +h & \text{if } n_1 = n_0 + 1 \\ -h & \text{if } n_1 = n_0 - 1 \end{cases}$$

assume

$$\delta x_{nm} = -\delta x_{mn}$$

$$\delta x_{nn} = 0$$

if $f = 0 \Rightarrow W(\mu_0) = 0$

$\Rightarrow \overline{W} = 0 \Rightarrow \dot{I} = 0$ ESS when $f \neq 0$: $\overline{W} = -f \langle \delta x \rangle$

$$= -f \sum_{n_1 n_0} \mathcal{P}(\mu_0)\, \delta x_{n_1 n_0}$$

$$= -f \sum_{n_1 n_0} W_{n_1 n_0} p^s_{n_0}\, \delta x_{n_1 n_0}$$

LDB: $$\frac{W_{n_1 n_0}}{W_{n_0 n_1}} = e^{-\beta(E_{n_1} - E_{n_0} - f\,\delta x_{n_1 n_0})}$$

$\Rightarrow$ elements of W depend on f

$\Rightarrow$ stationary state $\vec{p}^{\,s}$ (right e-vec of W w/ e-val 1, $W\vec{p}^{\,s} = \vec{p}^{\,s}$) also depends on f

in limit $f \to 0$ ESS: $p_n^s \xrightarrow[f\to 0]{} p_n^{eq} = \frac{e^{-\beta E_n}}{Z}$

$W_{nm} \xrightarrow[f\to 0]{} W_{nm}^{eq}$

expand for each n, m:

$$\underbrace{W_{nm} p_m^s}_{g(f)} = W_{nm}^{eq} p_m^{eq} \left[1 + c_{nm} f + \ldots \right]$$

$\uparrow$ some number related to Taylor exp.

$$g(f) = g(0) + g'(0) f + \frac{1}{2} g''(0) f^2 + \ldots$$

$$= g(0) \left[1 + \frac{g'(0)}{g(0)} f + \frac{1}{2} \frac{g''(0)}{g(0)} f^2 + \ldots \right]$$

plug into expression for avg. work $\bar{W}$:

$$\bar{W} = -f \sum_{n_1 n_0} \underbrace{W^{eq}_{n_1 n_0} p^{eq}_{n_0}}_{\text{symm. under } n_0 \leftrightarrow n_1} \delta x_{n_1 n_0}$$

anti-symm. under $n_0 \leftrightarrow n_1$

1st term $= 0$

$$- f \sum_{n_1 n_0} W^{eq}_{n_1 n_0} p^{eq}_{n_0} c_{n_1 n_0} f \, \delta x_{n_1 n_0} + \cdots$$

LDB in equil: $\frac{W^{eq}_{n_1 n_0}}{W^{eq}_{n_0 n_1}} = e^{-\beta(E_{n_1} - E_{n_0})} = \frac{p^{eq}_{n_1}}{p^{eq}_{n_0}}$

true for every n_1, n_0 $\Rightarrow$ $W^{eq}_{n_1 n_0} p^{eq}_{n_0} = W^{eq}_{n_0 n_1} p^{eq}_{n_1}$

Work $\bar{W} = -f \langle \delta x \rangle$

$$= -f^2 \underbrace{\sum_{n_1 n_0} W^{eq}_{n_1 n_0} p^{eq}_{n_0} c_{n_1 n_0} \delta x_{n_1 n_0}}_{\ell \text{ some number}}$$

$$\langle \delta x \rangle = f \ell$$

$$\bar{W} = -f^2 \ell$$

$$\bar{I} = -\frac{\bar{W}}{T} = \frac{f^2 \ell}{T} \geq 0 \Rightarrow \ell \geq 0 \text{ to satisfy 2nd law of thermod.}$$

$$\sigma = \frac{\bar{I}}{\delta t} = \text{entropy prod. rate}$$

$$\frac{\langle \delta x \rangle}{\delta t} \equiv J \quad \text{thermodynamic "flux"}$$

$$\frac{f}{T} \equiv \phi \quad \text{thermodynamic "force"}$$

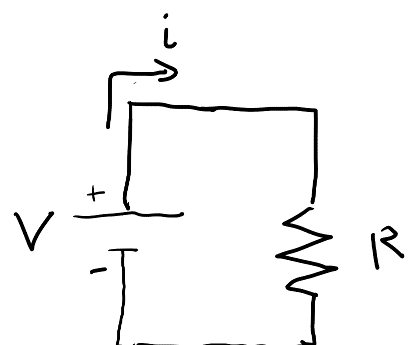
rewrite: $\sigma = J\phi = L\phi^2 \qquad L \equiv \frac{\ell T}{\delta t}$

$J = L\phi \qquad = \text{constant} \geq 0$

Onsager coefficient

L describes response of a sys. (flux) to applied thermo. force ϕ

example: stationary state of an electrical circuit



$$i = \frac{1}{R} V \qquad J = \underbrace{\frac{T}{R}}_{L} \underbrace{\frac{V}{T}}_{\phi}$$

power dissipated $\quad -\frac{W}{\delta t} = iV$

heat ↘ $Q = W = -TI$

2nd law for NESS

$$\frac{Q}{\delta t} = \frac{W}{\delta t} = \frac{-TI}{\delta t}$$

$$-\frac{1}{T}\frac{W}{\delta t} = \frac{I}{\delta t} = \sigma$$

$$\underbrace{i}_{J}\underbrace{\frac{V}{T}}_{\phi} = \sigma$$

Summary:

$J = i \qquad L = \frac{T}{R} \geq 0$

$\phi = \frac{V}{T} \qquad \Rightarrow R \geq 0$ 2nd law