$$I(\nu) = -k_B \ln\left[\frac{1+\sum_{\alpha=1}^{M} f_\alpha b_{n_1}^{(\alpha)}}{1+\sum_\alpha f_\alpha b_{n_0}^{(\alpha)}}\right] + \frac{1}{T}\sum_\alpha f_\alpha\, \delta x_{n_1 n_0}^{(\alpha)}$$

$\nu = (n_1, n_0)$

simplify notation: $I(\nu) = \frac{1}{T}\sum_\alpha f_\alpha\, \delta y_{n_1 n_0}^{(\alpha)}$

in small f_α limit: $\delta y_{n_1 n_0}^{(\alpha)} \approx \delta x_{n_1 n_0}^{(\alpha)} - k_B T\left(b_{n_1}^{(\alpha)} - b_{n_0}^{(\alpha)}\right)$

$\ln(1+\epsilon) \approx \epsilon$

$\phi_\alpha = \frac{f_\alpha}{T} \Rightarrow I(\nu) = \sum_\alpha \phi_\alpha\, \delta y_{n_1 n_0}^{(\alpha)}$

Eq. 1: $\underbrace{\frac{\langle I(\nu)\rangle}{\delta t}}_{\sigma} \approx \frac{1}{2k_B}\frac{\langle I^2(\nu)\rangle}{\delta t}$ from IFT

$$\sigma = \frac{1}{2k_B\,\delta t}\left\langle \sum_{\alpha\gamma} \phi_\alpha \phi_\gamma\, \delta y^{(\alpha)} \delta y^{(\gamma)}\right\rangle$$

$$\sigma = \sum_{\alpha\gamma} \phi_\alpha \phi_\gamma L_{\alpha\gamma}$$

$\sigma = \vec{\phi}^T L \vec{\phi}$

$$\Rightarrow \quad L_{\alpha\gamma} = \frac{1}{2k_B\,\delta t}\left\langle \delta y^{(\alpha)} \delta y^{(\gamma)}\right\rangle \quad \text{[Eq. 3]}$$

recipe for calculating $L_{\alpha\gamma}$

$\Rightarrow$ shows that $L_{\alpha\gamma} = L_{\gamma\alpha}$ for any $\alpha \neq \gamma$

$\Rightarrow$ Onsager reciprocity [1931 Nobel prize] two loop circuit:

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad L = \begin{pmatrix} \frac{T}{R} + \frac{T}{R'} & -\frac{T}{R'} \\ -\frac{T}{R'} & \frac{T}{R'} \end{pmatrix}$$

write out Eq. 3 in detail:

$$L_{\alpha\gamma} = \frac{1}{2k_B \delta t} \sum_{n, n_0} \underbrace{W_{n,n_0} p_{n_0}^{s}}_{\mathcal{J}(\mu_0)} \delta y_{n,n_0}^{(\alpha)} \delta y_{n,n_0}^{(\gamma)}$$

$$W_{n,n_0} p_{n_0}^{s} = W_{n,n_0}^{eq} p_{n_0}^{eq} \left(1 + \sum_\alpha f_\alpha C_{n,n_0}^{(\alpha)} + \dots \right)$$

$$L_{\alpha\gamma} \approx \frac{1}{2k_B \delta t} \sum_{n, n_0} W_{n,n_0}^{eq} p_{n_0}^{eq} \delta y_{n,n_0}^{(\alpha)} \delta y_{n,n_0}^{(\gamma)} + \dots$$

$$L_{\alpha\gamma} \approx \frac{1}{2k_B \delta t} \langle \delta y^{(\alpha)} \delta y^{(\gamma)} \rangle_{eq}$$

avg. in ESS

fluctuation-dissipation theorem [FDT]
(Callen, Welton, Kubo, etc. 1950's)

RHS: can be evaluated using equil. trajectories (fluctuations in equil.)

LHS: $L_{\alpha\gamma}$ tell you about entropy prod.

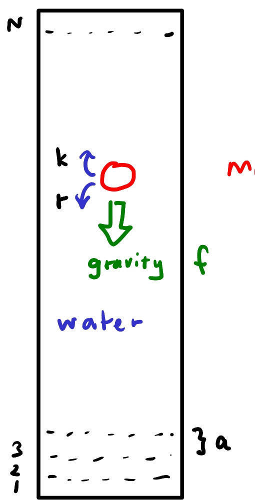
$$\sigma = \vec{\phi}^{T} L \vec{\phi} = \frac{I}{\delta t} = \frac{-Q}{\delta t}$$

heat per unit time dissipated

dissipation

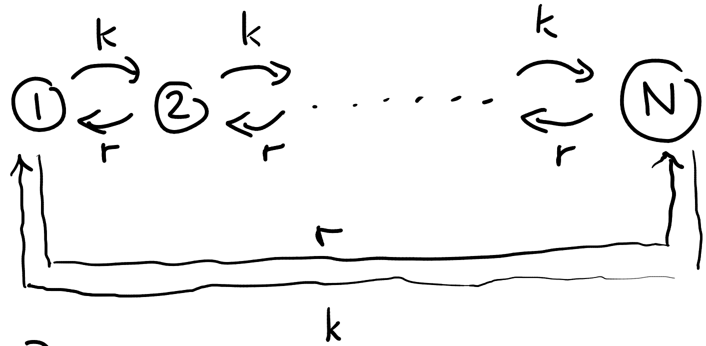
into environ. when out of equilibrium

Example: Brownian motion



micron-sized bead

for math simplicity assume periodic BC



don't matter if cylinder is big + bead starts in middle

E_n = const. for all n

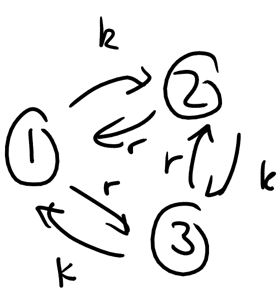
LDB: $$\frac{k}{r} = e^{-\beta(\underbrace{E_{n+1}-E_n}_{0} + fa)}$$

$$= e^{-\beta f a}$$

coupling to one force $(M=1)$

stat. state $p_n^s(f) = \frac{1}{N} = p_n^{eq}$

special case where $p_n^s(f)$ is f indep. by symmetry



$$W\vec{p}^s = \vec{p}^s \Rightarrow \vec{p}^s = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$p_n^s(f) = p_n^{eq}\left(1 + \cancel{b_n}^{\,0} f + \cdots\right)$$

$$\Rightarrow \quad \delta y_{n_1 n_0} = \delta x_{n_1 n_0} = \begin{cases} a & \text{if } n_1 = n_0 + 1 \\ -a & \text{if } n_1 = n_0 - 1 \end{cases}$$

L is 1×1 matrix (L is scalar)

FDT: $\quad 2 k_B \, \delta t \, L = \langle \delta x^2 \rangle_{eq}$

$$\frac{\langle \delta x \rangle}{\delta t} = J = L \phi = L \frac{f}{T}$$

current

mean vel. of bead

from fluid mechanics $\quad J = \frac{1}{\gamma} f$

γ drag friction coeff.

$\frac{L}{T} = \frac{1}{\gamma} \quad \Rightarrow$ rewrite FDT

$$\langle \delta x^2 \rangle_{eq} = 2 k_B L \, \delta t$$

$$= 2 \underbrace{\frac{k_B T}{\gamma}}_{D} \delta t$$

D diffusion const.

$$\boxed{D = \frac{k_B T}{\gamma}}$$

Einstein 1905