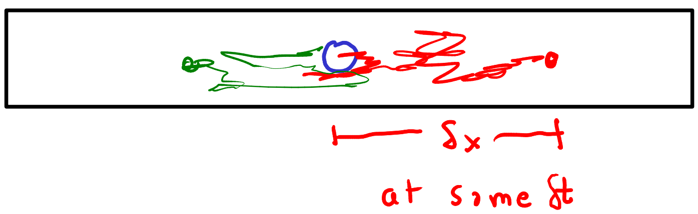
Einstein 1905: $D = \frac{k_B T}{\gamma}$

$J = \frac{1}{\gamma} f$ = mean vel. of bead

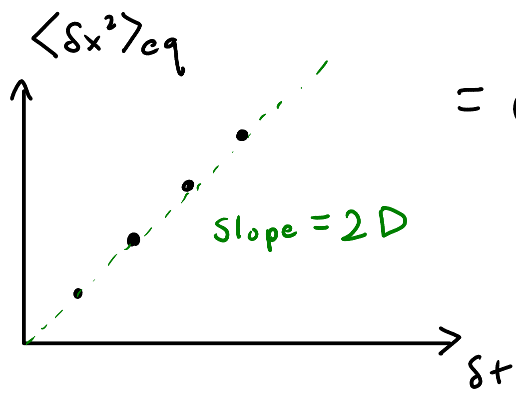
$$\langle \delta x^2 \rangle_{eq} = 2 D \delta t$$

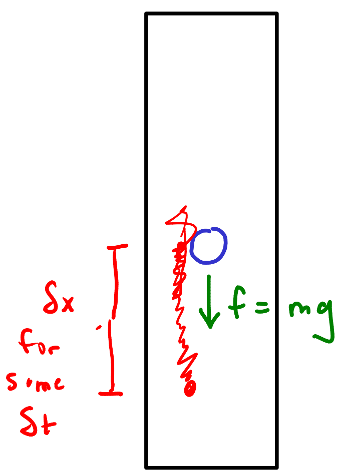
equil. setup: $f = 0$ (along horiz. axis where we measure δx)



$\langle \delta x \rangle_{eq}$ = avg. over many runs $= 0$

$$\langle \delta x^2 \rangle_{eq} = 2 D \delta t$$





$$\frac{\langle \delta x \rangle}{\delta t} = \frac{1}{\gamma} f$$

γ = drag friction coefficient

alternative: use 19th century fluid mechanics (Stokes):

$$\gamma = 6 \pi \eta R$$

η = viscosity of fluid

R = bead radius

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very similar example: resistive electrical circuits

J = current

f = voltage

$\phi = \frac{f}{T}$

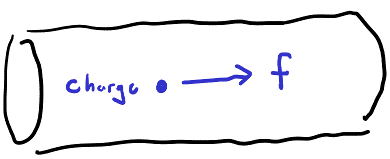
$L = \frac{T}{R}$ R is resistance

$J = L\phi$

FDT: $\langle \delta x^2 \rangle_{eq} = 2 k_B \delta t \, L$

$$\frac{\langle \delta x \rangle}{\delta t} = J = L\phi = L\frac{f}{T}$$

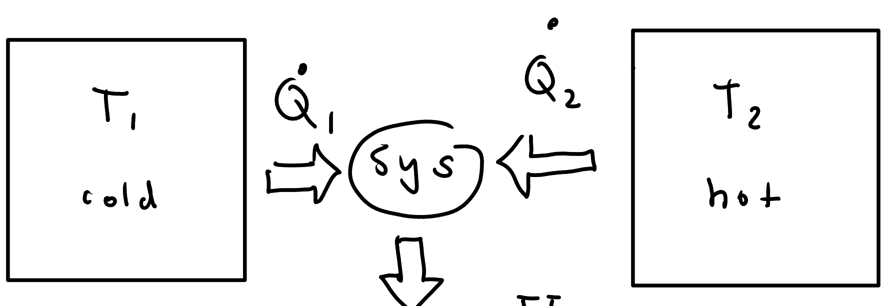
Ohm's law



$$\langle \delta x^2 \rangle_{eq} = 2 \frac{k_B T}{R} \delta t$$

random motion of charges when $f = 0$ (zero voltage) } Johnson–Nyquist noise [1926]

Multiple driving forces:



$$P = \frac{W}{\delta t}$$

stationary state (or cycle):

(1) $P = \dot{Q}_1 + \dot{Q}_2$

(2) $\sigma = \frac{I}{\delta t} = -\frac{\dot{Q}_1}{T_1} - \frac{\dot{Q}_2}{T_2} \geq 0$

$$\dot{Q}_1 = P - \dot{Q}_2$$

$$\sigma = \dot{Q}_2\left(\frac{1}{T_1} - \frac{1}{T_2}\right) - \frac{P}{T_1} \geq 0$$

apply linear thermo; write this as

$$\sigma = \vec{J} \cdot \vec{\phi}$$

assume we are close to equil: $T_1 = T$, $T_2 = T + \Delta T$, $\Delta T > 0$, $\frac{\Delta T}{T} \ll 1$

$$\frac{1}{T_1} - \frac{1}{T_2} \approx \frac{\Delta T}{T^2}$$ Taylor series in ΔT

$$\overline{W} = f \langle \delta x \rangle$$

$$P = \frac{\overline{W}}{\delta t} = f \frac{\langle \delta x \rangle}{\delta t} \equiv f \dot{x}$$

$$\sigma = \underbrace{\dot{Q}_2}_{\substack{J_H \\ \text{heat} \\ \text{current}}} \underbrace{\frac{\Delta T}{T^2}}_{\substack{\phi_H \\ \text{heat} \\ \text{"force"}}} - \underbrace{\dot{x}}_{\substack{\text{"work} \\ \text{current"} \\ J_W}} \underbrace{\frac{f}{T}}_{\substack{\text{force} \\ \text{coupled} \\ \text{to} \\ \text{work} \\ \phi_W}} \geq 0$$

$$\sigma = \vec{J} \cdot \vec{\phi} \qquad \vec{J} = \begin{pmatrix} J_H \\ J_W \end{pmatrix} \qquad \vec{\phi} = \begin{pmatrix} \phi_H \\ \phi_W \end{pmatrix}$$

$$\vec{J} = L \vec{\phi} \Rightarrow \begin{pmatrix} \dot{Q}_2 \\ -\dot{x} \end{pmatrix} = \underbrace{\begin{pmatrix} L_{HH} & L_{HW} \\ L_{WH} & L_{WW} \end{pmatrix}}_{\text{Onsager matrix}} \begin{pmatrix} \frac{\Delta T}{T^2} \\ \frac{f}{T} \end{pmatrix}$$

starting point $\Rightarrow$ derive efficiency at maximal power

[van der Broeck PRL 2005]

What do we know about L?

1) symmetric : $L_{WH} = L_{HW}$

$\Rightarrow$ eigenvalues are real: λ_1 , λ_2

$\Rightarrow$ e-vecs are orthonormal: $\vec{u}^{(i)} \cdot \vec{u}^{(j)} = \delta_{ij}$

$i = 1, 2$

2) we know that $\sigma = \vec{\phi}^T L \vec{\phi} \geq 0$ for any $\vec{\phi}$ (small)

plug in $\vec{\phi} = \varepsilon \vec{u}^{(i)}$

$\uparrow$ small coeff.

$$\sigma = \varepsilon^2 \vec{u}^{(i)T} L \vec{u}^{(i)} = \varepsilon^2 \lambda_i \underbrace{\vec{u}^{(i)T} \vec{u}^{(i)}}_{1}$$

$$= \varepsilon^2 \lambda_i \geq 0$$

$$\Rightarrow \lambda_i \geq 0$$

$\Rightarrow$ all e-vals $\lambda_i \geq 0$

3) $\det L = \lambda_1 \lambda_2 \geq 0$

4) plug in $\vec{\phi} = \varepsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow \sigma = \vec{\phi}^T L \vec{\phi} = \varepsilon^2 L_{ii} \geq 0 \Rightarrow L_{ii} \geq 0$

$\Rightarrow$ diag. elements of L are positive

Summary: $L_{HH}, L_{WW} \geq 0 \quad L_{WH} = L_{HW}$

$\det(L) = L_{HH} L_{WW} - L_{WH} L_{HW} \geq 0$

What is power in this system:

$$P = f\dot{x} = -f\left(L_{WH}\overset{\Delta T/T^2}{\phi_H} + L_{WW}\overset{f/T}{\phi_W}\right)$$

$$= -L_{WW}\frac{f^2}{T} - L_{WH}\frac{\Delta T f}{T^2}$$

$$= -\frac{L_{WW}}{T}\left(f + \frac{L_{WH}\Delta T}{2L_{WW}T}\right)^2 + \frac{L_{WH}^2\Delta T^2}{4L_{WW}T^3}$$