closed qu. system: $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$

$$\Rightarrow \boxed{\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{+}(t)}$$

$$\Rightarrow \quad S(\hat{\rho}(t)) = S(\hat{\rho}(0))$$

contrast: closed classical sys. which is ergodic + mixing (hockey stadium billiard)

Gibbs entropy $\quad S(t) = -\sum_{n=1}^{N} p_n(t) \ln p_n(t)$

inc. w/ time $\quad \xrightarrow[t\to\infty]{} \quad S^{eq} = k_B \ln N$

$$\underset{\text{von Neumann}}{S(\hat{\rho})} \quad \underset{\text{classical limit}}{\neq} \quad \text{Gibbs entropy}$$

("therm. entropy")

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next steps: add interactions w/ outside env.

$\Rightarrow$ $\hat{\rho}(t)$ generally evolves according to Choi-Krauss repr. theorem

$\Rightarrow$ qu. master eqn for $\hat{\rho}(t)$

Simple example of interactions: projective measurements (collapse sys. onto e-states)

Initial ensemble: $\quad$ $\underline{\text{frac.}}$ $\quad p_1 \quad p_2 \quad \cdots$

$$\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

$\underline{\text{states}}$. $\quad |\psi_1\rangle \quad |\psi_2\rangle \quad \cdots$

measure $\hat{A}$

w/ e-states

$$\hat{A}|a\rangle = a|a\rangle$$

collapse to one of the $|a\rangle$ w/ prob. $|\langle a|\psi_1\rangle|^2$

" " " w/ prob. $|\langle a|\psi_2\rangle|^2$

final ensemble: $\quad$ $\underline{\text{frac.}}$ $\quad p_a \quad p_b \quad \cdots$

$\underline{\text{states}}$ $\quad |a\rangle \quad |b\rangle$

$$p_a = \sum_n p_n |\langle a|\psi_n\rangle|^2$$

final dens. oper:

$$\hat{\rho}' = \sum_a p_a |a\rangle\langle a|$$

$$= \sum_{a,n} p_n \langle a|\psi_n\rangle\langle\psi_n|a\rangle\, |a\rangle\langle a|$$

$$= \sum_a |a\rangle\langle a| \underbrace{\left[\sum_n p_n |\psi_n\rangle\langle\psi_n|\right]}_{\hat{\rho}} |a\rangle\langle a|$$

$\hat{P}_a = |a\rangle\langle a|$ proj. operator

$\hat{P}_a^\dagger = |a\rangle\langle a|$

$$\boxed{\hat{\rho}' = \sum_a \hat{P}_a \hat{\rho} \hat{P}_a^\dagger}$$

dynamics of $\hat{\rho}$ under measurement compare to: $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger$ isolated sys.

here we have: $\underbrace{\sum_a \hat{P}_a^\dagger \hat{P}_a}_{\sum_a |a\rangle\langle a|} = \hat{I}$

example: initially 100% in state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\hat{\rho} = |\psi_1\rangle\langle\psi_1|$$

initial entropy $S(\hat{\rho}) \Rightarrow$ write $\hat{\rho}$ in a diag. basis & find e-vals

basis: $|\psi_1\rangle,\ |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{e-vals} \quad \begin{array}{l} p_1 = 1 \\ p_2 = 0 \end{array}$$

$$S(\hat{\rho}) = -\sum_n p_n \ln p_n = 0 \qquad \text{pure state}$$

measurement: $|\psi_1\rangle \xrightarrow{50\%} |0\rangle$

w/ $|0\rangle, |1\rangle$ as e-states $\qquad |\psi_1\rangle \xrightarrow[50\%]{} |1\rangle$

collapse

final dens. oper. $\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

alternate $= |0\rangle\underbrace{\langle 0|\hat{\rho}|0\rangle}_{1/2}\langle 0| + |1\rangle\underbrace{\langle 1|\hat{\rho}|1\rangle}_{1/2}\langle 1|$

$S(\hat{\rho}') \Rightarrow$ diag. basis $\{|0\rangle, |1\rangle\}$

$$\hat{\rho}' = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \Rightarrow S(\hat{\rho}') = \ln 2$$

$$> S(\hat{\rho}) = 0$$

Statement of general theorem (Choi-Kraus rep. theorem):

most general interactions w/ outside world can be expressed as a transf. $\hat{\rho} \to \hat{\rho}'$ w/ following structure:

$$\hat{\rho}' = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger$$

number $\Gamma \leq N^2$

N = dim. of Hilbert space

($N=2$ for $|0\rangle, |1\rangle$)

operators: $\hat{M}_\gamma$ Kraus operators

satisfy: $\sum_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma = \hat{I}$ identity previous examples are special cases:

i) closed sys $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^{+}$

$$\Gamma = 1 \qquad \hat{M}_1 = \hat{U}$$

$$\sum_{\gamma} \hat{M}_{\gamma}^{+} \hat{M}_{\gamma} = \hat{U}^{+}\hat{U} = \hat{I}$$

ii) measurements: $\hat{\rho}' = \sum_{a} \hat{P}_a \hat{\rho} \hat{P}_a^{+}$

$$\Gamma = N \qquad \hat{M}_{\gamma} \Rightarrow \hat{P}_a = |a\rangle\langle a|$$

$$\sum_{\gamma} \hat{M}_{\gamma}^{+} \hat{M}_{\gamma} = \sum_{a} |a\rangle\underbrace{\langle a|a\rangle}_{1}\langle a|$$

$$= \sum_{a} |a\rangle\langle a| = \hat{I}$$

note: $\hat{M}_{\gamma}^{+} \neq \hat{M}_{\gamma}$ in general

$\hat{M}_{\gamma}$ not necessarily unitary

theorem is $\sum_{\gamma} \hat{M}_{\gamma}^{+} \hat{M}_{\gamma} = \hat{I}$

not $\sum_{\gamma} \hat{M}_{\gamma} \hat{M}_{\gamma}^{+} = \hat{I}$ ✗

one later example: $\hat{M}_{\gamma} \propto |1\rangle\langle 0|$ "jump"