Choi-Kraus proof: how to transform a valid $\hat{\rho}$ to another valid $\hat{\rho}'$

valid: $\hat{\rho}^\dagger = \hat{\rho}$ $\quad$ $tr(\hat{\rho}) = 1$ $\quad$ $\langle i|\hat{\rho}|i\rangle \geq 0$ in any basis

also demand transform be linear:

example:

2 diff. pure ensembles A, B

$$\hat{\rho}_A = |\psi_1\rangle\langle\psi_1| \xrightarrow{\text{time evol.}} \hat{\rho}_A' \text{ valid}$$
$$\hat{\rho}_B = |\psi_2\rangle\langle\psi_2| \rightsquigarrow \hat{\rho}_B'$$

third ensemble: C

frac. f of $|\psi_1\rangle$

+ frac. 1-f of $|\psi_2\rangle$

$$\hat{\rho}_C = f|\psi_1\rangle\langle\psi_1| + (1-f)|\psi_2\rangle\langle\psi_2|$$

$$\xrightarrow{\text{time evol.}} \hat{\rho}_C' = f\hat{\rho}_A' + (1-f)\hat{\rho}_B'$$

transf. under time evol. should preserve linear combo's of density operators

<u>PROOF</u>: write $\hat{\rho}$ and $\hat{\rho}'$ in some basis $\{|i\rangle\}$ of our N-dim Hilbert space

$$\hat{\rho} = \sum_{ij} \rho_{ij} |i\rangle\langle j| \qquad \rho_{ij} = \langle i|\hat{\rho}|j\rangle$$

$$\hat{\rho} = \begin{array}{cc} & j \\ i & \begin{pmatrix} - & \rho_{ij} & \\ & & \\ & & \end{pmatrix} \end{array}$$

$$\hat{\rho}' = \sum_{lk} \rho'_{lk} |l\rangle\langle k| \qquad \rho'_{lk} = \langle l|\hat{\rho}'|k\rangle$$

mapping $\hat{\rho} \Rightarrow \hat{\rho}'$ is a mapping from one matrix to another

b/c mapping is linear, we can express it as a tensor S:

$$\rho'_{lk} = \sum_{ij} \underbrace{S_{li;kj}}_{\substack{\text{4 index tensor} \\ \text{each element = number}}} \rho_{ij}$$

RHS is a linear combo of ρ_{ij} elements

multiply both sides by $|l\rangle\langle k|$ + sum over l,k

$$\hat{\rho}' = \sum_{lk} \rho'_{lk} |l\rangle\langle k| = \sum_{ij\,lk} S_{li;kj} \underbrace{\langle i|\hat{\rho}|j\rangle}_{\rho_{ij}} |l\rangle\langle k|$$

$$= \sum_{ij\,lk} S_{\underbrace{li}_{\alpha};\underbrace{kj}_{\beta}} \underbrace{|l\rangle\langle i|}_{\hat{T}_\alpha} \hat{\rho} \underbrace{|j\rangle\langle k|}_{\hat{T}_\beta^+}$$

$$\alpha = (\underset{1\cdots N}{l}, \underset{1\cdots N}{i}) \qquad \beta = (k,j)$$

$$\hat{T}_\alpha \equiv |l\rangle\langle i|$$

$$\hat{T}_\beta \equiv |k\rangle\langle j|$$

$1 = (1,1)$

$2 = (1,2)$

$\vdots$

$N^2 = (N,N)$

$$\Rightarrow \quad \hat{\rho}' = \sum_{\alpha\beta} S_{\alpha\beta}\, \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger \qquad (1)$$

most general linear transf. b/t operators

$S_{\alpha\beta}$ we can interpret as element of an $N^2 \times N^2$ dim matrix S

S	1 $(1,1)$	2 $(1,2)$
1 $(1,1)$	S_{11}	$S_{12} = S_{11;12}$
2 $(1,2)$		

next step: enforce $\hat{\rho}'$ is a valid dens. oper if $\hat{\rho}$ is a density oper.

i) $\hat{\rho}'^\dagger = \hat{\rho}' \Rightarrow \quad \hat{\rho}'^\dagger = \sum_{\alpha\beta} S^*_{\alpha\beta}\, \hat{\tau}_\beta \hat{\rho} \hat{\tau}_\alpha^\dagger \qquad \hat{\rho}^\dagger = \hat{\rho}$

transform dummy vars : $\alpha \to \beta \qquad \beta \to \alpha$

$$\Rightarrow \hat{\rho}'^\dagger = \sum_{\alpha\beta} S^*_{\beta\alpha}\, \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger$$

compare to: $\hat{\rho}' = \sum_{\alpha\beta} S_{\alpha\beta}\, \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^\dagger$

$$\Rightarrow S^*_{\beta\alpha} = S_{\alpha\beta}$$

S $N^2 \times N^2$ matrix is Hermitian

$\Rightarrow$ there must exist a matrix U that diagonalizes S:

$$U^{+} S U = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

λ_γ are e-vals of S

U: columns are e-vecs of S

$$U^{+} U = U U^{+} = I \underset{\text{equival.}}{\Rightarrow} S = U \Lambda U^{+}$$

$$S_{\alpha\beta} = \sum_{\gamma,\mu} U_{\alpha\gamma} \underbrace{\Lambda_{\gamma\mu}}_{\lambda_\gamma \delta_{\gamma\mu}} (U^{+})_{\mu\beta}$$

$$= \sum_{\gamma} U_{\alpha\gamma} \lambda_\gamma (U^{+})_{\gamma\beta}$$

$$= \sum_{\gamma} U_{\alpha\gamma} \lambda_\gamma U^{*}_{\beta\gamma}$$

plug into Eq. (1):

$$\hat{\rho}' = \sum_{\alpha\beta\gamma} \lambda_\gamma U_{\alpha\gamma} \hat{\tau}_\alpha \hat{\rho} \hat{\tau}_\beta^{+} U^{*}_{\beta\gamma}$$

$\lambda_\gamma = \epsilon_\gamma |\lambda_\gamma|$

define new operators: Krans operators

$$\hat{M}_\gamma \equiv \sum_\alpha U_{\alpha\gamma} \hat{\tau}_\alpha \sqrt{|\lambda_\gamma|}$$

$$\hat{M}_\gamma^{+} \equiv \sum_\alpha U^{*}_{\alpha\gamma} \hat{\tau}_\alpha^{+} \sqrt{|\lambda_\gamma|}$$

$$= \sum_\beta U^{*}_{\beta\gamma} \hat{\tau}_\beta^{+} \sqrt{|\lambda_\gamma|}$$

dummy $\alpha \to \beta$

$$\Rightarrow \quad \hat{\rho}' = \sum_{\gamma} \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \qquad \epsilon_\gamma = \frac{\lambda_\gamma}{|\lambda_\gamma|} = \pm 1$$

$$= \mathrm{sign}(\lambda_\gamma)$$

next step: enforce $\mathrm{tr}(\hat{\rho}') = 1$