$$\hat{\rho}' = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger \qquad \epsilon_\gamma = \pm 1$$

$\gamma = 1, \dots, N^2$

N = dim. of Hilb. space

ii) $$\mathrm{tr}(\hat{\rho}') = 1$$

$$= \sum_\gamma \epsilon_\gamma \, \mathrm{tr}(\hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^\dagger)$$

$$= \sum_\gamma \epsilon_\gamma \, \mathrm{tr}(\hat{M}_\gamma^\dagger \hat{M}_\gamma \hat{\rho})$$

$$= \mathrm{tr}\Big(\underbrace{\Big[\sum_\gamma \epsilon_\gamma \hat{M}_\gamma^\dagger \hat{M}_\gamma \Big]}_{\hat{A} \text{ some oper.}} \hat{\rho} \Big)$$

$\mathrm{tr}(\hat{A}\hat{B}\hat{C})$
$= \mathrm{tr}(\hat{C}\hat{A}\hat{B})$
$= \mathrm{tr}(\hat{B}\hat{C}\hat{A})$

$\mathrm{tr}(\hat{A}+\hat{B})$
$= \mathrm{tr}(\hat{A}) + \mathrm{tr}(\hat{B})$

$\hat{A}^\dagger = \hat{A}$ Hermitian

$$1 = \mathrm{tr}(\hat{A}\hat{\rho})$$

choose a basis where $\hat{A}$ is diag: $\{|m\rangle\}$
express trace in that basis

$$1 = \sum_m \langle m|\hat{A}\hat{\rho}|m\rangle$$

$$\sum_n |n\rangle\langle n| = \hat{I}$$

$$= \sum_{m,n} \underbrace{\langle m|\hat{A}|n\rangle}_{= A_{mm}\delta_{mn}} \langle n|\hat{\rho}|m\rangle$$

$$1 = \sum_m A_{mm} \rho_{mm}$$

also know: $1 = \sum_m \rho_{mm}$ $\qquad \rho_{mm} \geq 0 \quad \forall m$

true for any $\hat{\rho}$ $\Rightarrow$ only way this works is if $A_{mm} = 1 \quad \forall m$

in this basis $\hat{A} = \hat{I} = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma$

iii) in any basis $\rho'_{ii} = \langle i | \hat{\rho}' | i \rangle \geq 0$

$$\hat{\rho}' = \sum_\gamma \epsilon_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^+$$

$$\langle i | \hat{\rho}' | i \rangle = \sum_\gamma \epsilon_\gamma \langle i | \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^+ | i \rangle \geq 0$$

this has to be true for any valid $\hat{\rho}$

choose $\hat{\rho} = |\psi\rangle\langle\psi|$ any pure state

$$\langle i | \hat{\rho}' | i \rangle = \sum_\gamma \epsilon_\gamma \langle i | \hat{M}_\gamma | \psi \rangle \langle \psi | \hat{M}_\gamma^+ | i \rangle$$

$$= \sum_\gamma \epsilon_\gamma |\langle i | \hat{M}_\gamma | \psi \rangle|^2 \geq 0$$

Since this true for any $|\psi\rangle$

$\Rightarrow$ guarantees $\epsilon_\gamma = +1$ for all γ

Choi-Kraus (CK) : $\boxed{\hat{\rho}' = \sum_\gamma \hat{M}_\gamma \hat{\rho} \hat{M}_\gamma^+ \quad \text{where} \quad \sum_\gamma \hat{M}_\gamma^+ \hat{M}_\gamma = \hat{I}}$

note: set of up to N^2 $\hat{M}_\gamma$ operators is not unique master equ: $$\vec{p}(t+\delta t) = W \vec{p}(t)$$

$$\approx$$

$$\vec{p}(t) + \frac{d\vec{p}}{dt}\delta t = W \vec{p}(t)$$

W columns sum to 1

$$\frac{d\vec{p}}{dt} = \frac{(W - \hat{I})}{\delta t}\vec{p}(t)$$

Ω columns sum to zero

$$\frac{d\vec{p}}{dt} = \Omega \vec{p}(t)$$

GOAL: $$\frac{\partial}{\partial t}\hat{\rho}(t) = \sim\sim\sim$$

CK theorem applies at every time step:

(1) $$\hat{\rho}(t+\delta t) = \sum_{\gamma=1}^{\Gamma} \hat{M}_\gamma \hat{\rho}(t) \hat{M}_\gamma^{+} \qquad \Gamma \leq N^2$$

assumption: dynamics is Markovian

$\Rightarrow$ $\hat{\rho}(t+\delta t)$ depends only on $\hat{\rho}(t)$

try to do: $$\hat{\rho}(t+\delta t) = \hat{\rho}(t) + \sim\sim\, \delta t$$

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \frac{\hat{\rho}(t+\delta t) - \hat{\rho}(t)}{\delta t} = \sim\sim$$

easy case: no interactions w/ env.

$$\hat{\rho}(t+\delta t) = \hat{U}_s \hat{\rho}(t) \hat{U}_s^\dagger$$

$\Gamma = 1$

$\hat{U}_s = e^{-i\hat{\mathcal{H}}_s \delta t/\hbar}$

$\hat{U}_s^\dagger \hat{U}_s = \hat{I}$

sys. Hamiltonian time-ind: $\hat{\mathcal{H}}_s$

$$\hat{M}_1 = \hat{U}_s \qquad \hat{M}_\gamma = 0 \qquad \gamma > 1$$

$$\hat{\rho}(t+\delta t) = \left(\hat{I} - \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \ldots\right) \hat{\rho}(t) \left(\hat{I} + \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \ldots\right)$$

$$= \hat{\rho}(t) - \frac{i}{\hbar}\left[\hat{\mathcal{H}}_s, \hat{\rho}(t)\right] \delta t + \ldots$$

$$\Rightarrow \boxed{\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{\mathcal{H}}_s, \hat{\rho}\right]}$$

qu. master eqn for isolated sys. (von Neumann sys.)

turn on interactions:

CK matrices: $\hat{M}_1 = \hat{I} - \frac{i\hat{\mathcal{H}}_s \delta t}{\hbar} + \underbrace{\hat{K}\,\delta t}_{\text{correction due to interact. w/ env.}}$

$\gamma = 1$

$N^2 - 1$ remaining matrices for $\gamma > 1$: $\hat{M}_\gamma = \sqrt{\delta t}\, \hat{L}_\gamma$ for some oper. $\hat{L}_\gamma$

$\hat{L}_\gamma, \hat{K}$ describe interactions w/ env.

to turn off inter., $\hat{L}_\gamma, \hat{K} \to 0$

demand: $\hat{I} = \sum_{\gamma} \hat{M}_\gamma^\dagger \hat{M}_\gamma$

$$= \hat{I} + \delta t \underbrace{\left[2\hat{K} + \sum_{\gamma>1} \hat{L}_\gamma \hat{L}_\gamma \right]}_{=0} + \ldots$$

higher order in δt

$$\Rightarrow \hat{K} = -\frac{1}{2} \sum_{\gamma>1} \hat{L}_\gamma^\dagger \hat{L}_\gamma$$

plug into Eq. (1):

algebra:

$$\boxed{\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} &= -\frac{i}{\hbar} [\hat{H}_s, \hat{\rho}] \\ &+ \sum_{\gamma>1} \left(\hat{L}_\gamma \hat{\rho} \hat{L}_\gamma^\dagger - \frac{1}{2} \hat{L}_\gamma^\dagger \hat{L}_\gamma \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_\gamma^\dagger \hat{L}_\gamma \right) \end{aligned}}$$

qu. master eq.

Lindblad eq. (1976)

due to environ. interactions