master equ: $p_n(t_i) \Rightarrow \vec{p}(t_i) = \begin{pmatrix} p_1(t_i) \\ p_2(t_i) \\ \vdots \\ p_N(t_i) \end{pmatrix}$

$$\vec{p}(t_{i+1}) = W(t_i)\,\vec{p}(t_i)$$

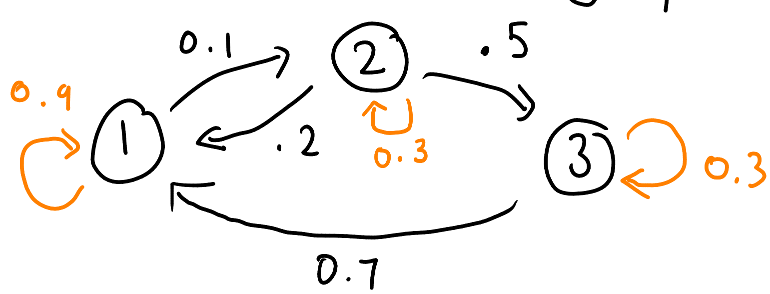
$\downarrow$

col's sum to 1

can be represented as

a transition graph

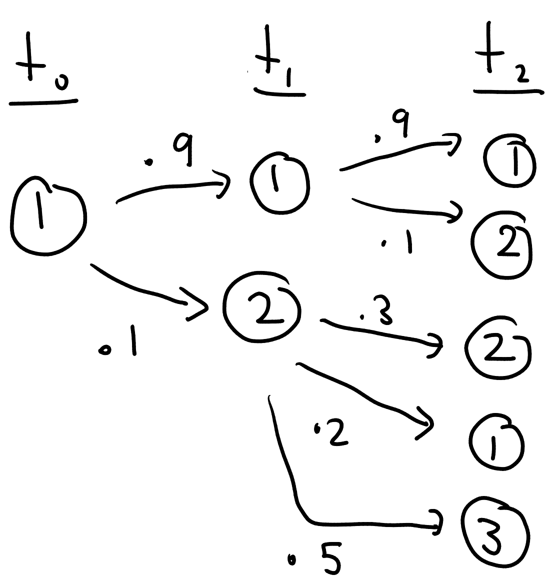
graph:



recall $\mathcal{P}(v) = W_{n_i n_{i-1}}(t_{i-1}) \cdots W_{n_1 n_0}(t_0)\, p_{n_0}(t_0)$

$v = (n_0, n_1, n_2, \ldots, n_i)$

all possible 3 time step traj. starting at 1:



v	$\mathcal{P}(v)$
1 1 1	.81
1 1 2	.09
1 2 2	.03
1 2 1	.02
1 2 3	.05

sum : $\overline{1}$

$$\vec{p}(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{p}(t_1) = \begin{pmatrix} .9 \\ .1 \\ 0 \end{pmatrix} = W \vec{p}(t_0)$$

$$\vec{p}(t_2) = \begin{pmatrix} .83 \\ .12 \\ .05 \end{pmatrix} = W \vec{p}(t_1)$$

so far: $p_n(t_i)$

n: discrete state (DS)

t_i: discrete time (DT)

DS DT

$$\vec{p}(t_{i+1}) = W(t_i)\vec{p}(t_i)$$

generalizations: continuous time (CT)

$t_i \to t$ continuous

δt is small

continuous state (CS)

$n \to x$ continuous

	DT	CT
DS	DTDS master eqn. Simulate trans. graph	CTDS master eqn. Kinetic Monte Carlo (Gillespie)
CS	numerical simulations	Fokker-Planck eqn. Langevin eqn.

black = prob. picture
red = traj. picture

assume: time-indep. envir. conditions

$$W(t_i) = W$$

sol'n: $\vec{p}(t_n) = W^n \vec{p}(t_0)$

observation: W^n often converges to a const. matrix as $n \to \infty$

Why?

if $W^n \to$ const. as $n \to \infty$ $\Longrightarrow$ $\vec{p}(t_{i+1}) = W^{i+1} \vec{p}(t_0)$

$$= W \underbrace{W^i \vec{p}(t_0)}_{\vec{p}(t_i)}$$

as $i \to \infty$

$W^{i+1} \approx W^i$

$\vec{p}(t_{i+1}) \approx \vec{p}(t_i) \equiv \vec{p}^{\,s}$ Station. probability

Stat. probability right e-vec of W w/ e-val 1

$$\boxed{\vec{p}^{\,s} = W \vec{p}^{\,s}}$$

Questions:

1) does this e-vec always exist?

2) is it unique?

3) $W^n \vec{p}(t_0)$ as $n \to \infty$: is this guaranteed to approach $\vec{p}^s$?

Q1: does W always have at least one e-vec w/ e-val $\underline{1}$?

quick lemma: matrix M

right e-vec: $M\vec{v} = \lambda \vec{v}$

λ satisfies:

$$\det(M - \lambda I) = 0$$

left e-vec: $\vec{u}^T M = \sigma \vec{u}^T$

transpose: $M^T \vec{u} = \sigma \vec{u}$

$\Rightarrow$ σ is a sol'n of

$$\det(M^T - \sigma I) = 0$$

$$\det((M - \sigma I)^T) = 0$$

$$\det(M - \sigma I) = 0$$

$\det A^T = \det A$

left e-vals are <u>same</u> as right e-vals if we know a left e-val 6 exists $\Rightarrow$ there must be a right e-vec w/

$$M\vec{v} = 6\vec{v}$$

$M = W \Rightarrow$ show there always exists a left e-vec $\vec{u}^T = (1\ 1\ 1 \cdots 1)$ w/ e-val 1

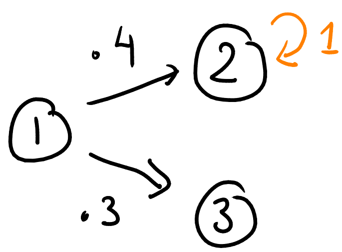
$$\vec{u}^T W = \vec{u}$$

always true b/c col's of W sum to 1

$$(1\ 1\ 1)\begin{pmatrix} .1 & .2 & .1 \\ .3 & .5 & .1 \\ .6 & .3 & .8 \end{pmatrix} = (1\ 1\ 1)$$

$\Rightarrow$ there has to be at least one $\vec{p}^{\,s}$ w/ e-val 1: $W\vec{p}^{\,s} = \vec{p}^{\,s}$

example:



$$W = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & .3 & 0 & 0 \\ 2 & .4 & 1 & 0 \\ 3 & .3 & 0 & 1 \end{array}$$

start

$$p_A^s = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad p_B^s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$0 \le \alpha \le 1$

$$p_C^s = \alpha\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1-\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \alpha \\ 1-\alpha \end{pmatrix}$$