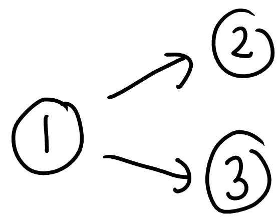
$$\vec{p}(t_{i+1}) = W\,\vec{p}(t_i)$$

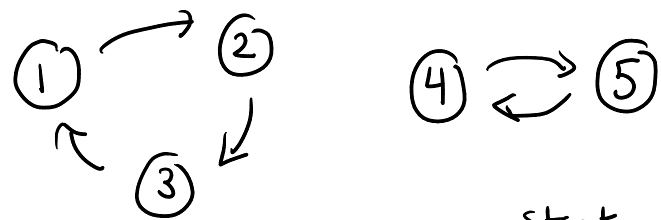
$$\vec{p}^{\,s} = W\,\vec{p}^{\,s}$$

example #1:



$\infty$ of $\vec{p}^{\,s}$

example #2:



$$\begin{array}{cc} & \begin{array}{cc} 1\ 2\ 3 & 4\ 5 \end{array}\quad \text{start} \\ W = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{end}\quad 5 \end{array} & \left(\begin{array}{c|c} \phantom{0} & 0 \\ \hline 0 & \phantom{0} \end{array}\right) \end{array}$$

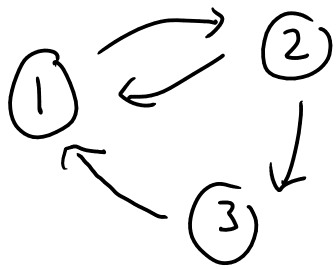
$$\vec{p}_A^{\,s} = \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ 0 \\ 0 \end{pmatrix} \qquad \vec{p}_B^{\,s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bullet \\ \bullet \end{pmatrix}$$

$$\vec{p}_C^{\,s} = \alpha\,\vec{p}_A^{\,s} + (1-\alpha)\,\vec{p}_B^{\,s} \qquad 0 \le \alpha \le 1$$

Restrict focus to systems w/ unique $\vec{p}^{\,s}$:

- always consider connected graphs
  (avoid ex. #2)
- demand our graph is strongly connected: if we start at any state, we can reach any other state following arrows

(exclude ex. # 1)



strongly connected

not micro-reversible

also known:

ergodic graph

as $t \to \infty$ you will visit all states

- final condition: (will prove for classical + quantum systems)

microscopic reversibility (MR)

if $W_{ij} \neq 0 \Rightarrow W_{ji} \neq 0$

$j \to i$ arrow exists $\qquad$ $i \to j$ arrow exists

$W_{ji} = 0 \Rightarrow W_{ij} = 0$

either no arrow or double arrow b/t any two states

categories:

$$\text{all graphs} \begin{cases} \to \text{connected} \begin{cases} \to \text{ergodic} \begin{cases} \to \boxed{\text{MR}} \\ \to \text{non-MR} \end{cases} \\ \to \text{non-ergodic} \end{cases} \\ \to \text{disconnected} \end{cases}$$

next steps:
- prove MR graphs have unique $\vec{p}^{\,s}$
- explore why physics is MR

first need to introduce: "mean hitting time" of a graph

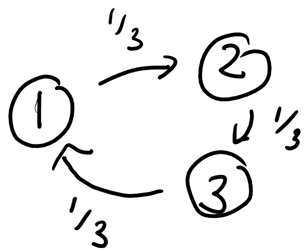
MHT

"first passage time"

MHT = avg. # of time steps $\delta t$ to get to state $j$ given start at $i$

$\equiv h_{ji}$ ($< \infty$ for ergodic graphs)

example:




trajectory: $\underbrace{1 \quad 1}_{} \;\; \overbrace{1 \quad 2}^{\text{length } 2} \quad 2 \quad 3 \quad 3 \quad 3 \quad \underbrace{1 \quad 2}_{\text{len. } 1} \quad 3$

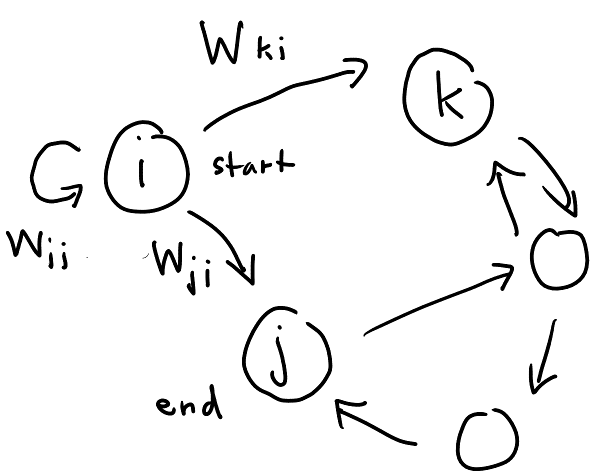
len. 1

$h_{21}$ = avg. of all trip lengths from $1 \to 2$

$h_{11}$ = " " " $1 \to 1$

$N^2$ quantities $h_{ij}$ to calculate

$\Rightarrow N^2$ equations?



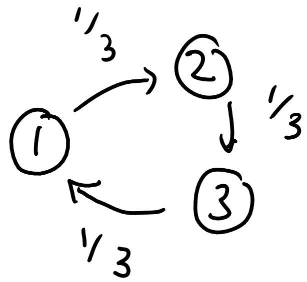

$$h_{ji} = W_{ji} \cdot \underset{\substack{\uparrow \\ \text{trip of len. } 1}}{1} + \sum_{k \neq j} W_{ki} (1 + h_{jk})$$

$$h_{ji} = \underbrace{W_{ji} + \sum_{k \neq j} W_{ki}}_{\sum_k W_{ki} = 1} + \sum_{k \neq j} W_{ki} h_{jk}$$

$$\Rightarrow \quad \boxed{h_{ji} = 1 + \sum_{k \neq j} h_{jk} W_{ki}} \qquad \begin{matrix} i = 1, \ldots, N \\ j = 1, \ldots, N \end{matrix}$$

Eq. $*$ $\qquad$ $N^2$ equations for $N^2$ unknowns

example:




$$W = \begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} \end{array}$$

$i=1,\ j=1$: $\quad h_{11} = 1 + \sum_{k \neq 1} h_{1k} W_{k1} = 1 + h_{12} W_{21}$

$i=2,\ j=1$: $\quad h_{12} = 1 + h_{12} W_{22} + h_{13} W_{32}$

$i=3,\ j=1$: $\quad h_{13} = 1 + h_{13} W_{33}$ $\qquad$ etc.

$$\Rightarrow \quad h_{13} = 3 \qquad h_{12} = 6 \qquad h_{11} = 3$$

Why is this useful?

1) given a $W$, choose any $\vec{p}^{\,s}$ (at least one exists)

2) multiply Eq. $*$ by $p_i^s$

$$p_i h_{ji} = p_i^s \left(1 + \sum_{k \neq j} h_{jk} W_{ki}\right)$$

3) sum both sides over $i$:

$$\sum_i p_i h_{ji} = 1 + \sum_i p_i^s \sum_{k \neq j} h_{jk} W_{ki}$$

$$p_j^s h_{jj} + \sum_{i \neq j} p_i^s h_{ji} = 1 + \sum_{k \neq j} h_{jk} \underbrace{\sum_i W_{ki} \, p_i^s}_{p_k^s}$$

$$W \vec{p}^{\,s} = \vec{p}^{\,s}$$

$$p_j^s h_{jj} + \sum_{i \neq j} p_i^s h_{ji} = 1 + \sum_{k \neq j} p_k^s h_{jk}$$

same

$$p_j^s h_{jj} = 1 \Rightarrow \quad p_j^s = \frac{1}{h_{jj}} > 0$$

b/c $h_{jj} < \infty$ in an ergodic net

$\Rightarrow$ unique solution for $\vec{p}^{\,s}$ in an ergodic net.