RG Methods in Statistical Field Theory: Problem Set 6 Solution

Problem 1

In this problem we investigate the nature of the singularities in the Gaussian model as $T \to T_c^+$ ($r \to 0^+$). Even though at $r = 0$ the system exhibits fluctuations at all length scales, we will show that the singularities are caused entirely by long-wavelength fluctuations (small $\mathbf{q}$ modes).

(a) Consider the d-dimensional Gaussian model, written in terms of Fourier-transformed variables $\mathbf{m}(\mathbf{q})$, where $\mathbf{m}$ is the n-component order parameter:

$$\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(r + cq^2 + Lq^4 + \cdots \right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{H} \cdot \mathbf{m}(\mathbf{q} = 0)$$

Here $\mathbf{H} = H\hat{\mathbf{e}}_1$ is a uniform magnetic field pointing along the $\hat{\mathbf{e}}_1$ axis. Using the facts about Gaussian functional integrals discussed earlier in class, find the exact expression for the partition function Z of this system. Show that the free energy per volume f can be written as:

$$f = -\frac{1}{\beta V} \ln Z = \frac{n}{2\beta} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \ln \left[v_0^{-1} \beta (r + cq^2 + Lq^4 + \cdots) \right] - \frac{H^2}{2r}$$

Hint: Depending on how you calculate Z, you might end up with a factor of $\delta^{(d)}(\mathbf{q} = 0)$ in one of the terms. You can find the value of this factor using the definition: $(2\pi)^d \delta^{(d)}(\mathbf{q}) = \int dx \, \exp(i\mathbf{q} \cdot \mathbf{x})$. Thus $\delta^{(d)}(0) = V/(2\pi)^d$, where V is the volume of the system.

Answer: The partition function $Z = \int \mathcal{D}\mathbf{m} \, \exp(-\beta\mathcal{H})$ has the form of a Gaussian functional integral with kernel $K(q) = \beta(r + cq^2 + Lq^4 + \cdots)$ and external field $h_i(\mathbf{q}) = \beta H_i (2\pi)^d \delta^{(d)}(\mathbf{q})$ for each component $i = 1, \ldots, n$ of the order parameter $m_i(\mathbf{q})$. Thus the solution is:

$$\begin{aligned}
Z &= \left(\frac{(2\pi)^\infty}{\det K} \right)^{n/2} \exp\left(\frac{1}{2} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{h_i(-\mathbf{q}) h_i(\mathbf{q})}{K(\mathbf{q})} \right) \\
&= \left(\frac{(2\pi)^\infty}{\det K} \right)^{n/2} \exp\left(\frac{1}{2} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\beta^2 H^2 (2\pi)^d \delta^{(d)}(-\mathbf{q}) (2\pi)^d \delta^{(d)}(\mathbf{q})}{\beta(r + cq^2 + Lq^4 + \cdots)} \right) \\
&= \left(\frac{(2\pi)^\infty}{\det K} \right)^{n/2} \exp\left(\frac{\beta H^2 (2\pi)^d \delta^{(d)}(0)}{2r} \right) = \left(\frac{(2\pi)^\infty}{\det K} \right)^{n/2} \exp\left(\frac{\beta H^2 V}{2r} \right)
\end{aligned}$$

where:

$$\ln(\det K) = V \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left(v_0^{-1} K(\mathbf{q}) \right)$$

We find that:

$$f = -\frac{1}{\beta V} \ln Z = \frac{n}{2\beta} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \ln \left[v_0^{-1} \beta (r + cq^2 + Lq^4 + \cdots) \right] - \frac{H^2}{2r}$$

(b) Let us look at the magnetic susceptibility, $\chi = -\partial^2 f/\partial H^2$ evaluated at $H = 0$. Show that $\chi \propto r^{-1}$, so it diverges as $r \to 0^+$. Note that this divergence is entirely due to the $\mathbf{H} \cdot \mathbf{m}(\mathbf{q} = 0)$ term in the Hamiltonian $\mathcal{H}$, where the magnetic field couples to the $\mathbf{q} = 0$ mode (infinite wavelength fluctuation). The singularity does not depend in any way on the cutoff Λ. If we change the cutoff, adding or subtracting high $\mathbf{q}$ modes in the Hamiltonian, the singular behavior of χ is not affected.

Answer: From the expression for f from part (a):

$$\chi = -\frac{\partial^2 f}{\partial H^2} = \frac{1}{2} r^{-1}$$

(c) Calculate the leading behavior of the specific heat for small r at $H = 0$, $C \approx -T_c \partial^2 f/\partial r^2$. Show that it can be written as:

$$C \approx A \int_0^\Lambda dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2}$$

where the constant $A = nk_B T_c^2 S_d/2(2\pi)^d$ and S_d is the area of a d-dimensional unit sphere. Argue that for $d > d_c$, there is no divergence in C as $r \to 0^+$. Find d_c.

Answer: The leading behavior of derivatives of f with respect to r can be found by treating $\beta \approx 1/k_B T_c$ as a constant. At $H = 0$ we have:

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{nk_B T_c}{2} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{r + cq^2 + Lq^4 + \cdots} \\
&= \frac{nk_B T_c S_d}{2(2\pi)^d} \int_0^\Lambda dq \, \frac{q^{d-1}}{r + cq^2 + Lq^4 + \cdots} \\
C \approx -T_c \frac{\partial^2 f}{\partial r^2} &= \frac{nk_B T_c^2 S_d}{2(2\pi)^d} \int_0^\Lambda dq \, \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2}
\end{aligned}$$

The integral is bounded from above by $q = \Lambda$, so the divergence can only come from the lower limit at $q = 0$. At $r = 0$, $q \to 0^+$, we can approximate the denominator $(r+cq^2+Lq^4+\cdots)^2 \approx c^2q^4$, so the integrand $\propto q^{d-5}$. Thus if $d > d_c = 4$ the integral is convergent.

(d) Now consider the case $d < d_c$. Let us break up the integral into two parts, one going from $q = 0$ to Λ/b, and the other from $q = \Lambda/b$ to Λ:

$$C \approx A \int_0^{\Lambda/b} dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2} + A \int_{\Lambda/b}^{\Lambda} dq \frac{q^{d-1}}{(r + cq^2 + Lq^4 + \cdots)^2} \equiv C_< + C_>$$

Argue that for any $b > 1$, the contribution $C_>$ must be finite in the limit $r \to 0^+$.

Answer: The integral in the contribution $C_>$ is bounded from above by Λ, and from below by Λ/b. Since at $r = 0$ the integrand does not blow up in the range $\Lambda/b < q < \Lambda$, $C_>$ must be finite.

(e) The result of part (d) means that the divergence in C is entirely contained in the $C_<$ term. Show that as $r \to 0^+$, $C_< \approx Br^{-\alpha}$, where B is a constant independent of Λ and b. Find the exponent α. *Hint:* Non-dimensionalize the $C_<$ integral using the variable $x = (c/r)^{1/2}q$.

Answer: Making the substitution $x = (c/r)^{1/2}q$, we have:

$$
\begin{aligned}
C_< &= \frac{Ar^{d/2}}{c^{d/2}} \int_0^{(c/r)^{1/2}\Lambda/b} dx \frac{x^{d-1}}{(r + rx^2 + \frac{Lr^2}{c^2}x^4 + \cdots)^2} \\
&= \frac{Ar^{d/2-2}}{c^{d/2}} \int_0^{(c/r)^{1/2}\Lambda/b} dx \frac{x^{d-1}}{(1 + x^2 + \frac{Lr}{c^2}x^4 + \cdots)^2}
\end{aligned}
$$

In the limit $r \to 0^+$ the upper bound of the integral goes to ∞, and we find:

$$
C \approx \frac{Ar^{d/2-2}}{c^{d/2}} \int_0^{\infty} dx \frac{x^{d-1}}{(1+x^2)^2}
$$

For $d < 4$ the integral here converges to a constant independent of Λ and b. Thus we have $C \propto r^{-\alpha}$, with $\alpha = 2 - d/2$.

Note that parts (d) and (e) are true for any $b > 1$, even in the limit $b \gg \Lambda$, where $C_<$ corresponds to an integral over a tiny ball of radius Λ/b surrounding $\mathbf{q} = 0$ in the Brillouin zone. Thus the small $\mathbf{q}$ modes determine the divergence in the specific heat. The cutoff Λ, or any other details of the high $\mathbf{q}$ behavior, have no affect on the singularity.

Problem 2

Up to now we have only considered systems with short-range interactions. In magnetic lattice models we had a nearest-neighbor spin-spin interaction, and in the continuum limit this gave us derivative terms like $(\nabla \mathbf{m}(\mathbf{x}))^2$ in the Landau-Ginzburg Hamiltonian. But real physical systems can also have long-range effects, decaying slowly with distance, like magnetic dipole-dipole interactions. How would such interactions affect the critical behavior? In this problem we look at this question in the context of the Gaussian model.

(a) Let us add a long-range interaction $\mathcal{H}_{LD}$ to the Hamiltonian of the d-dimensional Gaussian model, where:

$$
\mathcal{H}_{LD} = \int d^d\mathbf{x} \int d^d\mathbf{y} J(|\mathbf{x} - \mathbf{y}|)\mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{y})
$$

and $J(r) = A/r^{d+\sigma}$ for some constants $A, \sigma > 0$. Show that in terms of Fourier modes, this interaction can be written as:

$$
\mathcal{H}_{LD} = K_\sigma \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, q^\sigma \mathbf{m}(\mathbf{q}) \cdot \mathbf{m}(-\mathbf{q})
$$

where K_σ is a constant which depends on the value of σ. *Hint:* It is useful to change variables to $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$ and $\mathbf{r} = (\mathbf{x} - \mathbf{y})/2$. There will be an integral over $\mathbf{r}$ from which the $\mathbf{q}$

dependence can be factored out using the substitution $\mathbf{s} = q\mathbf{r}$. The constant K_σ involves an integral (independent of $\mathbf{q}$) which you do *not* need to evaluate.

Answer:

$$\begin{aligned}
\mathcal{H}_{LD} &= A \int d^d\mathbf{x} d^d\mathbf{y}\, \frac{\mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}} \\
&= \frac{A}{2^{d+\sigma}} \int d^d\mathbf{R}\, d^d\mathbf{r}\, \frac{\mathbf{m}(\mathbf{R}+\mathbf{r}) \cdot \mathbf{m}(\mathbf{R}-\mathbf{r})}{r^{d+\sigma}} \\
&= \frac{A}{2^{d+\sigma}} \int d^d\mathbf{R}\, d^d\mathbf{r}\, \frac{1}{r^{d+\sigma}} \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \frac{d^d\mathbf{q}_2}{(2\pi)^d}\, \mathbf{m}(\mathbf{q}_1) \cdot \mathbf{m}(\mathbf{q}_2) e^{i\mathbf{q}_1\cdot(\mathbf{R}+\mathbf{r}) + i\mathbf{q}_2\cdot(\mathbf{R}-\mathbf{r})} \\
&= \frac{A}{2^{d+\sigma}} \int d^d\mathbf{r}\, \frac{1}{r^{d+\sigma}} \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \frac{d^d\mathbf{q}_2}{(2\pi)^d}\, \mathbf{m}(\mathbf{q}_1) \cdot \mathbf{m}(\mathbf{q}_2) e^{i\mathbf{r}\cdot(\mathbf{q}_1-\mathbf{q}_2)} (2\pi)^d \delta^{(d)}(\mathbf{q}_1+\mathbf{q}_2) \\
&= \frac{A}{2^{d+\sigma}} \int d^d\mathbf{r}\, \frac{1}{r^{d+\sigma}} \int \frac{d^d\mathbf{q}_1}{(2\pi)^d}\, \mathbf{m}(\mathbf{q}_1) \cdot \mathbf{m}(-\mathbf{q}_1) e^{2i\mathbf{r}\cdot\mathbf{q}_1} \\
&= \frac{A}{2^{d+\sigma}} \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, \mathbf{m}(\mathbf{q}) \cdot \mathbf{m}(-\mathbf{q}) \int d^d\mathbf{r}\, \frac{e^{2i\mathbf{r}\cdot\mathbf{q}}}{r^{d+\sigma}}
\end{aligned}$$

The $\mathbf{r}$ integral we can simplify through the substitution $\mathbf{s} = q\mathbf{r}$:

$$\mathcal{H}_{LD} = \frac{A}{2^{d+\sigma}} \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, \mathbf{m}(\mathbf{q}) \cdot \mathbf{m}(-\mathbf{q})\, q^\sigma \int d^d\mathbf{s}\, \frac{e^{2i\mathbf{s}\cdot\hat{\mathbf{q}}}}{s^{d+\sigma}}$$

Here $\hat{\mathbf{q}}$ is the unit vector in the direction of $\mathbf{q}$, but the $\mathbf{s}$ integral gives the same answer for all $\mathbf{q}$ (because we are integrating over the entire volume). Thus we can write:

$$\mathcal{H}_{LD} = \frac{K_\sigma}{2} \int \frac{d^d\mathbf{q}}{(2\pi)^d}\, q^\sigma \mathbf{m}(\mathbf{q}) \cdot \mathbf{m}(-\mathbf{q}) \qquad \text{where} \quad K_\sigma \equiv \frac{A}{2^{d+\sigma-1}} \int d^d\mathbf{s}\, \frac{e^{2i\mathbf{s}\cdot\hat{\mathbf{q}}}}{s^{d+\sigma}}$$

(b) Thus the Gaussian model with the long-range interaction has the form:

$$\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(r + K_\sigma q^\sigma + cq^2 + Lq^4 + \cdots\right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{H} \cdot \mathbf{m}(\mathbf{q}=0)$$

Construct a renormalization-group transformation for this system, and find equations for r', K'_σ, c', L', ... Leave the equations in terms of the parameter ζ, where ζ is the constant of proportionality in the definition $\mathbf{m}'(\mathbf{q}') = \zeta^{-1}\mathbf{m}_<(\mathbf{q})$. (Do not choose a particular value for ζ just yet.)

Answer: Following the same steps as for the Gaussian model in class, we can write $\mathcal{H}$ as a sum of slow mode and fast mode parts: $\mathcal{H} = \mathcal{H}_< + \mathcal{H}_>$. Integrating out the fast modes just gives an overall constant factor multiplying Z, and our effective Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}_<$:

$$\tilde{\mathcal{H}} = \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(r + K_\sigma q^\sigma + cq^2 + Lq^4 + \cdots\right) |\mathbf{m}_<(\mathbf{q})|^2 - \mathbf{H} \cdot \mathbf{m}_<(\mathbf{q}=0)$$

Making the substitutions $\mathbf{q}' = b\mathbf{q}$ and $\mathbf{m}'(\mathbf{q}') = \zeta^{-1}\mathbf{m}_<(\mathbf{q})$ we find:

$$\begin{aligned}\tilde{\mathcal{H}}' &= b^{-d}\zeta^2 \int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{2}\left(r + K_\sigma b^{-\sigma} q'^\sigma + cb^{-2}q'^2 + Lb^{-4}q'^4 + \cdots\right)|\mathbf{m}'(\mathbf{q})|^2 - \zeta\mathbf{H}\cdot\mathbf{m}'(\mathbf{q}'=0) \\ &= \int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{2}\left(r' + K'_\sigma q'^\sigma + c' q'^2 + L' q'^4 + \cdots\right)|\mathbf{m}'(\mathbf{q})|^2 - \mathbf{H}'\cdot\mathbf{m}'(\mathbf{q}'=0)\end{aligned}$$

where:

$$r' = \zeta^2 b^{-d} r, \quad K'_\sigma = \zeta^2 b^{-d-\sigma}K_\sigma, \quad c' = \zeta^2 b^{-d-2}c, \quad L' = \zeta^2 b^{-d-4}L, \quad \ldots \quad H' = \zeta H$$

(c) Consider the case where $\sigma > 2$, $c > 0$, and $K_\sigma, L, \ldots$ have arbitrary values. Choose an appropriate ζ, and show that the long-range interaction is irrelevant at the fixed point: it does not affect the critical behavior of the system.

<u>Answer:</u> In this case we would like to fix $c' = c$, so $\zeta = b^{(d+2)/2}$ and the fixed point is at $r^* = K_\sigma^* = L^* = \cdots = H^* = 0$, $c^* \neq 0$. The RG equation for K_σ becomes: $K'_\sigma = b^{2-\sigma}K_\sigma$. Since $\sigma > 2$, the long-distance interaction is irrelevant at the fixed point.

(d) Consider the case where $\sigma < 2$, $K_\sigma > 0$, and $c, L, \ldots$ have arbitrary values. Choose an appropriate ζ, and calculate the critical exponents γ, ν, and η. You should find that some of the exponents in this case depend on σ. Thus if the decay of the long-range interaction is sufficiently slow ($\sigma < 2$), it affects the critical behavior of the system.

<u>Answer:</u> In this case we would like to fix $K'_\sigma = K_\sigma$, so $\zeta = b^{(d+\sigma)/2}$ and the fixed point is at $r^* = c^* = L^* = \cdots = H^* = 0$, $K_\sigma^* \neq 0$. We find the RG equations for r and H, giving the thermal and magnetic eigenvalues y_T and y_H at the fixed point:

$$r' = b^\sigma r \quad \Rightarrow \quad y_T = \sigma, \qquad H' = b^{(d+\sigma)/2}H \quad \Rightarrow \quad y_H = (d+\sigma)/2$$

Using the same analysis as in class, we can express the exponents γ, ν, and η in terms of y_T and y_H:

$$\gamma = \frac{2y_H - d}{y_T} = 1, \qquad \nu = \frac{1}{y_T} = \frac{1}{\sigma}, \qquad \eta = d - 2y_H + 2 = 2 - \sigma$$