RG Methods in Statistical Field Theory: Problem Set 8

due: Friday, November 24, 2006

In class we saw that second-order RG generated additional terms in the Landau-Ginzburg Hamiltonian, and I claimed that all these terms were irrelevant. To try to understand this, let us consider a d-dimensional system with an n-component order parameter $\mathbf{m}(\mathbf{x})$, described by a general Hamiltonian $\mathcal{H} = \mathcal{H}_0 + U$, where the Gaussian part $\mathcal{H}_0$ is given by:

$$\mathcal{H}_0 = \int d^d\mathbf{x}\, \frac{1}{2} \left(rm^2 + c(\nabla m)^2 + L(\nabla^2 m)^2 + \cdots \right)$$

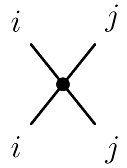
and the non-Gaussian perturbation has the form:

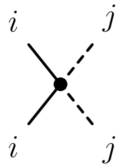
$$U = \int d^d\mathbf{x} \, \left(um^4 + vm^2(\nabla m)^2 + \cdots u_6 m^6 + \cdots + u_8 m^8 + \cdots \right)$$

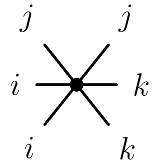
Here $m^{2k} \equiv (\mathbf{m}\cdot\mathbf{m})^k$ and $(\nabla m)^2 = \partial_i m_j \partial_i m_j$. Formally there are an infinite number of parameters, but we will concentrate on a represenative few: r, c, L, u, v, u_6, u_8. It will be easy to generalize the results for these terms to any other term. The Fourier-transformed Hamiltonian becomes:

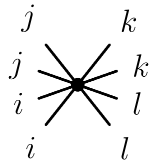
$$\begin{aligned}
\mathcal{H}_0 &= \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left(r + cq^2 + Lq^4 + \cdots \right) m_i(\mathbf{q}) m_i(-\mathbf{q}) \\
U &= u \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{q}_4}{(2\pi)^d}\, m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(\mathbf{q}_4) (2\pi)^d \delta^{(d)}(\mathbf{q}_1 + \cdots + \mathbf{q}_4) \\
&\quad + v \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{q}_4}{(2\pi)^d}\, (\mathbf{q}_3 \cdot \mathbf{q}_4) m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(\mathbf{q}_4) (2\pi)^d \delta^{(d)}(\mathbf{q}_1 + \cdots + \mathbf{q}_4) \\
&\quad + u_6 \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{q}_6}{(2\pi)^d}\, m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) \cdots m_k(\mathbf{q}_5) m_k(\mathbf{q}_6) (2\pi)^d \delta^{(d)}(\mathbf{q}_1 + \cdots + \mathbf{q}_6) \\
&\quad + u_8 \int \frac{d^d\mathbf{q}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{q}_8}{(2\pi)^d}\, m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) \cdots m_l(\mathbf{q}_7) m_l(\mathbf{q}_8) (2\pi)^d \delta^{(d)}(\mathbf{q}_1 + \cdots + \mathbf{q}_8) \\
&\quad + \cdots
\end{aligned}$$

We will represent the terms in U through the following vertex diagrams (we leave off the $\mathbf{q}$ labels for simplicity):

u: 

v: 

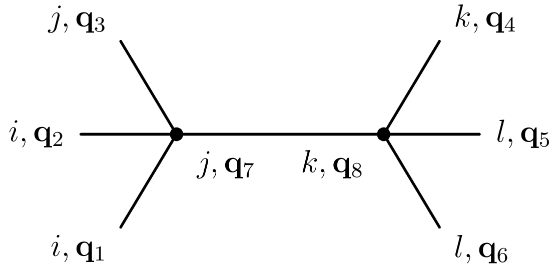
u_6: 

u_8: 

The dashed legs in the v vertex distinguish the $\mathbf{q}_3$ and $\mathbf{q}_4$ modes, which have a $(\mathbf{q}_3 \cdot \mathbf{q}_4)$ factor in the integral.

We construct the RG transformation in the standard way shown in class: the Gaussian Hamiltonian $\mathcal{H}_0$ can be separated into slow mode and fast mode portions, $\mathcal{H}_0 = \mathcal{H}_{0<} + \mathcal{H}_{0>}$, but this cannot be done for U, which mixes fast and slow modes. We get an effective Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}_{0<} - \frac{1}{\beta}\ln\langle e^{-\beta U}\rangle_{0>}$, with new parameters $\tilde{r}$, $\tilde{c}$, $\tilde{L}$, $\tilde{u}$, $\tilde{v}$, $\tilde{u}_6$, $\tilde{u}_8$, ... Before we look at the RG flows of these parameters, let us first derive a basic fact about diagrams.

(a) Consider the following diagram at order $\mathcal{O}(u^2)$, which appears to contribute to $\tilde{u}_6$:



Write down the integral for this diagram (do not worry about any numerical prefactors from the multiplicity or the cumulant expansion), and argue that the contribution to $\tilde{u}_6$ is actually zero. *Hint:* Remember that if there is some part of the integral that depends on the values of the slow mode momenta $\mathbf{q}_1, \ldots, \mathbf{q}_6$, it is only the zero-th order term of the corresponding Taylor expansion that contributes to $\tilde{u}_6$. When looking at the result, think carefully about the range of integration for the various $\mathbf{q}$.

This diagram is part of a larger class called *one-particle reducible* diagrams, all of which have the property that they can be divided into two disconnected parts by cutting a single internal line. Part (a) should convince you that for any such diagram, momentum conservation means that it has no contribution at zero-th order in its Taylor series (i.e. when all the external momenta are set to zero).

(b) Now we will consider the parameters $\tilde{r}$, $\tilde{c}$, $\tilde{L}$, $\tilde{u}$, $\tilde{v}$, $\tilde{u}_6$, and $\tilde{u}_8$ individually. For any given parameter K, the equation for $\tilde{K}$ will have the form:

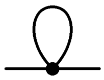
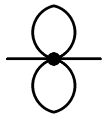
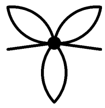
$$\tilde{K} = K + \mathcal{O}(\cdots)$$

We would like to know the order represented by $\mathcal{O}(\cdots)$. Thus we ask the question: for each parameter, what is the lowest order in the cumulant expansion of $-\frac{1}{\beta}\langle e^{-\beta U}\rangle_{0>}$ that we get a nonzero contribution to the effective Hamiltonian $\tilde{\mathcal{H}}$ (other than the contribution K)? What types of diagrams are responsible for the nonzero contribution at this order? (Draw the basic shapes, leaving out momenta and index labels; construct diagrams only from the vertices we showed above; do not count the multiplicities or evaluate any of the diagrams.) To give you a sense of what I want, here is the answer for $\tilde{r}$:

<u>Sample answer:</u> The first nonzero contribution to $\tilde{r}$ other than r comes from the first order part of the cumulant expansion. This contribution consists of diagrams of orders $\mathcal{O}(u)$, $\mathcal{O}(v)$, $\mathcal{O}(u_6)$, and $\mathcal{O}(u_8)$. We write the equation for $\tilde{r}$ as:

$$\tilde{r} = r + \mathcal{O}(u, v, u_6, u_8)$$

The nonzero contributing diagram types are:

$\mathcal{O}(u)$:  $\mathcal{O}(v)$:  $\mathcal{O}(u_6)$:  $\mathcal{O}(u_8)$: 

Do a similar analysis for each of the other parameters: $\tilde{c}$, $\tilde{L}$, $\tilde{u}$, $\tilde{v}$, $\tilde{u}_6$, and $\tilde{u}_8$.

(c) After substituting $\mathbf{q}' = b\mathbf{q}$ and $\mathbf{m}'(\mathbf{q}') = \zeta^{-1}\mathbf{m}(\mathbf{q})$ into the effective Hamiltonian, write down the RG equations for r', c', L', u', v', u_6', u_8', in terms of $\tilde{r}$, $\tilde{c}$, $\tilde{L}$, $\tilde{u}$, $\tilde{v}$, $\tilde{u}_6$, $\tilde{u}_8$. How do we choose the constant ζ? We would like to fix $c' = c$, and it is clear from part (b) that there are higher-order corrections, $\tilde{c} = c + \mathcal{O}(\cdots)$. Thus you should find that $\zeta = b^{(d+2)/2}(1 + \mathcal{O}(\cdots))$. Determine the order of the corrections to ζ. When we substitute ζ into the RG equations, these corrections will only be important at higher orders than we are interested in, so we can still use $\zeta \approx b^{(d+2)/2}$. Now we are ready to find the infinitesimal recursion relations. Let $b = 1 + \delta\ell$ and $d = 4 - \epsilon$, and show that the flow equations can be written as:

$$\begin{aligned}
\frac{dr}{d\ell} &= 2r + \mathcal{O}(u, v, u_6, u_8) \\
\frac{dL}{d\ell} &= -2L + \mathcal{O}(u^2, uv, v^2, \ldots) \\
\frac{du}{d\ell} &= \epsilon u - Bu^2 + \mathcal{O}(uv, v^2, \ldots) \\
\frac{dv}{d\ell} &= (-2 + \epsilon)v + \mathcal{O}(u^2, uv, v^2, \ldots) \\
\frac{du_6}{d\ell} &= (-2 + 2\epsilon)u_6 + \mathcal{O}(u_6^2, u_8^2, \ldots, u^3) \\
\frac{du_8}{d\ell} &= (-4 + 3\epsilon)u_8 + \mathcal{O}(u_6^2, u_8^2, \ldots, u^3)
\end{aligned}$$

Note that for simplicity not all the contributing combinations of u, v, u_6, u_8 are written explicitly in the $\mathcal{O}(\cdots)$ terms. There are also two special features which will be useful later: (i) the $\mathcal{O}(u^2)$ contribution to $du/d\ell$ is written directly as $-Bu^2$, which we know from class; (ii) for the $du_6/d\ell$ and $du_8/d\ell$ equations, we have included the $\mathcal{O}(u^3)$ contribution, even though it occurs at a higher order in the cumulant expansion. The reason for this will become apparent in part (e).

(d) The fixed point condition is obtained by setting all the flow equations from part (c) to zero. Show that the following is a fixed point solution:

$$r^* = L^* = u^* = v^* = u_6^* = u_8^* = 0$$

This is the Gaussian fixed point. Show that the eigenvalue exponents at the Gaussian fixed point are:

$$y_T = 2,\ y_L = -2,\ y_u = \epsilon,\ y_v = -2 + \epsilon,\ y_{u_6} = -2 + 2\epsilon,\ y_{u_8} = -4 + 3\epsilon$$

Thus for all $d > 4$ ($\epsilon < 0$) the only relevant direction is the thermal direction; all the others are irrelevant, meaning that a critical surface in the space of (L, u, v, u_6, u_8) flows to the Gaussian fixed point. For dimensions d just below 4 (small $\epsilon > 0$), the u direction becomes relevant, and thus for any nonzero u there will be flow away from the Gaussian fixed point, toward the Wilson-Fisher fixed point described in the next section.

(e) Assume that ϵ is small and positive. Show that there exists another fixed point, with $u^* \sim \mathcal{O}(\epsilon)$. Find the order in ϵ of the other parameters at the fixed point: r^*, L^*, v^*, u_6^*, and u_8^*. Besides u^*, what other parameters are $\mathcal{O}(\epsilon)$?

You can now relax; the problem set is over. Here is the payoff: from your answer in part (e), it should be clear that as $\epsilon \to 0$, the Wilson-Fisher fixed point moves continuously closer to the Gaussian fixed point, until the two merge at $\epsilon = 0$. The eigenvalue exponents of the Wilson-Fisher fixed point must also change continuously as $\epsilon \to 0$, approaching the values calculated for the Gaussian fixed point in part (d). Without doing any additional calculations, we can thus say that the eigenvalues at the Wilson-Fisher fixed point can differ from the Gaussian eigenvalues by at most $\mathcal{O}(\epsilon)$. If this is true, then using the answer of part (d) we can write down the following form for the eigenvalue exponents of the Wilson-Fisher fixed point:

$$y_T = 2 + \mathcal{O}(\epsilon),\ y_L = -2 + \mathcal{O}(\epsilon),\ y_u = \mathcal{O}(\epsilon),$$
$$y_v = -2 + \mathcal{O}(\epsilon),\ y_{u_6} = -2 + \mathcal{O}(\epsilon),\ y_{u_8} = -4 + \mathcal{O}(\epsilon)$$

Indeed when we explicitly calculated y_T and y_u in class, we found $y_T = 2 - \frac{n+2}{n+8}\epsilon$ and $y_u = -\epsilon$, which agree with the above form. Thus for dimensions just below 4 only the thermal direction is relevant; the Wilson-Fisher fixed point now controls the critical surface in (L, u, v, u_6, u_8) space. We can extend this analysis to all other higher order terms, and we will find the same answer. An infinite number of parameters are irrelevant at the fixed point, and the critical behavior is the same for a wide range of possible Hamiltonians (i.e. any Hamiltonian with $c > 0$ and nonzero u): this is one of the most dramatic examples of universality.