RG Methods in Statistical Field Theory:
Quiz 4 Solution

Friday, October 20, 2006

Consider a d-dimensional system with an $n = 1$ component order parameter $m(\mathbf{x})$. There is a mean-field solution m_0, and fluctuations away from the mean-field solution $m(\mathbf{x}) = m_0 + \phi(\mathbf{x})$ are described by the Fourier-transformed Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} c(q^2 + \xi^{-2})\phi(\mathbf{q})\phi(-\mathbf{q})$$

Here $c > 0$ is a constant, and ξ is the correlation length.

(a) We are interested in the correlation function for the fluctuations:

$$G(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x})\phi(\mathbf{x}')\rangle - \langle \phi(\mathbf{x})\rangle\langle \phi(\mathbf{x}')\rangle$$

Assume the system is translationally invariant, so that $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x} - \mathbf{x}')$. What is the Fourier-transformed correlation function $G(\mathbf{q})$ for the Hamiltonian above? (No long calculations are necessary. You can write it down by inspection.)

<u>**Answer:**</u> The partition function for this system is:

$$Z = e^{-\beta H_0}\int \mathcal{D}m(\mathbf{q}) e^{-\frac{\beta}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} c(q^2+\xi^{-2})\phi(\mathbf{q})\phi(-\mathbf{q})}$$

This has the Gaussian functional integral form discussed in class, with $K(\mathbf{q}) = \beta c(q^2 + \xi^{-2})$. Thus the correlation function is given by:

$$G(\mathbf{q}) = \frac{1}{K(\mathbf{q})} = \frac{1}{\beta c(q^2 + \xi^{-2})}$$

(b) Now do an inverse Fourier transform and write down an integral for the same-site correlation function $G(\mathbf{x}, \mathbf{x})$. (Do not evaluate the integral.) This measures the magnitude of fluctuations at a site $\mathbf{x}$. Show that for $d > 2$, we can approximate the integral and write

$$G(\mathbf{x}, \mathbf{x}) \approx \frac{S_d \Lambda^{d-2}}{\beta c(2\pi)^d(d-2)}$$

where S_d is the area of a d-dimensional unit sphere, and Λ is a large cutoff in q-space. *Hint:* Non-dimensionalize the integral using the variable $y \equiv q\xi$.

<u>**Answer:**</u>

$$G(\mathbf{x}, \mathbf{x}') = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} G(\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{\beta c(q^2 + \xi^{-2})}$$

$$\Rightarrow \quad G(\mathbf{x}, \mathbf{x}) = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{\beta c(q^2 + \xi^{-2})} = \frac{S_d}{(2\pi)^d}\int_0^\Lambda dq \frac{q^{d-1}}{\beta c(q^2 + \xi^{-2})}$$

Making the change of variables $y = q\xi$, and multiplying numerator and denominator by ξ^2, we find:

$$G(\mathbf{x}, \mathbf{x}) = \frac{S_d \xi^{2-d}}{\beta c(2\pi)^d}\int_0^{\Lambda\xi} dy \frac{y^{d-1}}{(y^2+1)} \approx \frac{S_d \xi^{2-d}}{\beta c(2\pi)^d}\frac{(\Lambda\xi)^{d-2}}{(d-2)} = \frac{S_d \Lambda^{d-2}}{\beta c(2\pi)^d(d-2)}$$