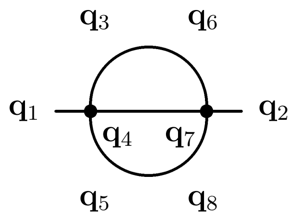
RG Methods in Statistical Field Theory: Quiz 7 Solution

Friday, November 24, 2006

We have a d-dimensional, $n = 1$ Landau-Ginzburg model, described by the Hamiltonian:

$$\begin{aligned}\mathcal{H} = &\int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{2}(r + cq^2)m(\mathbf{q})m(-\mathbf{q}) \\ &+ u\int_0^\Lambda \frac{d^d\mathbf{q}_1 \cdots d^d\mathbf{q}_4}{(2\pi)^{4d}}\, m(\mathbf{q}_1)m(\mathbf{q}_2)m(\mathbf{q}_3)m(\mathbf{q}_4)\,(2\pi)^d\delta^{(d)}(\mathbf{q}_1 + \cdots + \mathbf{q}_4)\end{aligned}$$

In working out the RG transformation for this model, consider the following $\mathcal{O}(u^2)$ diagram:



Note that the legs have no indices, since we are working with an $n = 1$ component model.

(a) What is the multiplicity M of this diagram?

__Answer:__ There are 4 ways of choosing a triplet of legs on the first vertex; there are 4 ways of choosing a triplet of legs on the second vertex. There are $3! = 6$ ways of connecting these two triplets together. Thus $M = 4 \times 4 \times 6 = 96$.

(b) The integral for this diagram has the form:

$$-\frac{\beta}{2}Mu^2\int_< \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2}{(2\pi)^{2d}}\, m_<(\mathbf{q}_1)m_<(\mathbf{q}_2)\, I(\mathbf{q}_1, \mathbf{q}_2)$$

Write down an integral expression for $I(\mathbf{q}_1, \mathbf{q}_2)$, but do not evaluate it.

__Answer:__

$$\begin{aligned}&I(\mathbf{q}_1, \mathbf{q}_2) \\ &= \int_> \frac{d^d\mathbf{q}_3 \cdots d^d\mathbf{q}_8}{(2\pi)^d}\frac{\delta(\mathbf{q}_3 + \mathbf{q}_6)\delta(\mathbf{q}_4 + \mathbf{q}_7)\delta(\mathbf{q}_5 + \mathbf{q}_8)}{(r + cq_3^2)(r + cq_4^2)(r + cq_5^2)}\delta(\mathbf{q}_1 + \mathbf{q}_3 + \mathbf{q}_4 + \mathbf{q}_5)\delta(\mathbf{q}_2 + \mathbf{q}_6 + \mathbf{q}_7 + \mathbf{q}_8) \\ &= \int_> \frac{d^d\mathbf{q}_3\, d^d\mathbf{q}_4\, d^d\mathbf{q}_5}{(2\pi)^d}\frac{\delta(\mathbf{q}_1 + \mathbf{q}_3 + \mathbf{q}_4 + \mathbf{q}_5)\delta(\mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4 - \mathbf{q}_5)}{(r + cq_3^2)(r + cq_4^2)(r + cq_5^2)}\end{aligned}$$

(c) Is the zeroth order term in the Taylor expansion of $I(\mathbf{q}_1, \mathbf{q}_2)$ nonzero? (Make an argument; do not actually calculate it.) If it is nonzero, to which term in the effective Hamiltonian $\tilde{\mathcal{H}}$ does it contribute?

__Answer:__ When $\mathbf{q}_1 = \mathbf{q}_2 = 0$, both delta functions in $I(\mathbf{q}_1, \mathbf{q}_2)$ become equal to: $\delta(\mathbf{q}_3 + \mathbf{q}_4 + \mathbf{q}_5)$. Even though $\mathbf{q}_3$, $\mathbf{q}_4$, and $\mathbf{q}_5$ are fast modes, it is possible to satisfy the condition $\mathbf{q}_3 + \mathbf{q}_4 + \mathbf{q}_5 = 0$. Hence the zeroth order term in the Taylor expansion is nonzero. Since there are two external legs, it contributes to the $\tilde{r}$ term in $\tilde{\mathcal{H}}$.